ON THE GEODESIC PANCYCLICITY OF MÖBIUS CUBES

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Abstract

For two vertices $X, Y \in V(G)$, a cycle is called a *geodesic cycle* with X and Y if a shortest path joining X and Y lies on the cycle. A graph G is called to be geodesic k-pancyclic if any two vertices X, Y on G have such geodesic cycle of length l that $2d_G(X,Y) + k \leq l \leq |V(G)|$. In this paper, we show that the n-dimensional Möbius cube MQ_n is geodesic 3-pancyclic for $n \geq 3$. This result is near optimal because there is no geodesic 1-cycle with two adjacent vertices in MQ_n.

Keywords: geodesic pancyclic, Möbius cubes, panconnected, pancyclic, shortest path.

1 Introduction

Interconnection networks are essential for parallel and distributed computing. A ring structure is often used as a interconnection architecture for local area network and as a control and data flow structure in distributed networks due to its good properties. To carry out a ring-structure algorithm on a specific multicomputer or a distributed system, the processes of the parallel algorithm need to be mapped to the nodes of the interconnection network in the system such that any two adjacent processes in the ring are mapped to two adjacent node of the network. For this purpose, it is desired that the targeted interconnection network posses a hamiltonian cycle, i.e., a cycle that passes every node of the network exactly once if the number of processes in the ring-structure parallel algorithm equals the number of nodes of the interconnection network. When the number of processes is less than the number of nodes of the network, the pancyclic property of the network with n nodes is desired, that is, there exists a cycle of length l in the network for each integer l with $4 \leq l \leq n$. The hypercube is one of the most popular interconnection networks since it has simple structure and is easy to implement.

Möbius cubes form a class of hypercube variants that give better performance with the same number of edges and vertices [2]. Cull et al. [2] proved that the n-dimensional Möbius cube, denoted by MQ_n, has several better properties than the n-dimensional hypercube, denoted by Q_n, for example, the diameter of MQ_n is about one half that of Q_n and graph embedding capability of MQ_n is better than Q_n.

With regard to the pancyclicity of Möbuis cubes, many related results have received considerable attention [3, 4, 5, 7, 10, 12, 13]. Fan [3] showed that an n-dimensional Möbius cube is pancyclic. Xu et al. [10] proved that an n-dimensional Möbius cube is edge-pancyclic, that is, every edge lies on a cycle of length l for each integer l with $4 \leq l \leq n$. Hu et al. [7] found that an n-dimensional Möbius cube is node-pancyclic, that is, every node lies on a cycle of length l for each integer l with $4 \leq l \leq n$. As concerns the fault-tolerant pancyclicity of Möbius cubes, Hsieh and Chen [4] proved that an n-dimensional Möbius cube with up to $n-2$ edge faults is pancyclic. After, Yang et al. [13] proposed that an n-dimensional Möbius cube is pancyclic in the presence of up to $n-2$ faulty nodes. When concerns pancyclicity of Möbius cubes in the presence of faulty nodes and/or edges, Yang et al. [12] proved that an n-dimensional Möbius cube is still pancyclic even if it has up to $n-2$ node and/or edge faults.

Here, we consider the geodesic cycle embedding problem that have been studied in [1, 6, 8] in Möbius cubes. In other words, for any two vertices, we want to find all the possible lengths of cycles including a shortest path joining them. A graph G is called geodesic k-pancyclic if any two vertices X, Y on G have such geodesic cycle of length l that $2d_G(X, Y) + k \leq l \leq |V(G)|$ where $d_G(X, Y)$ is the distance from X to Y in G. Hsu et al. [6] proved that an n-dimensional Augmented cube contains a geodesic pancyclic of length from $\max\{2d(X, Y), 3\} \leq l \leq 2^n$. Lai et al. [8] proposed that an n-dimensional Crossed cube is geodesic 4-pancyclic. In this paper, we prove that MQ_n is geodesic 3-pancyclic for $n \geq 3$.

This paper is organized as follows. In Section 2, we give some definitions and properties of Möbius cubes. In Section 3, we prove that MQ_n is geodesic 3-pancyclic. The final section concludes this papers.

2 Möbius cubes

Let the interconnection network be modeled by an undirected graph $G = (V, E)$ where the set of vertices $V(G)$ represents the processing elements of the network and the set of edges $E(G)$ represents the communication links. Throughout this paper, for the graph theoretic definitions and notations we follow [9]. Two vertices are adjacent when they are incident with a common edge. A path of length k from X to Y is a finite sequence of adjacent vertices written as $\langle X_1, X_2, \ldots, X_{k+1} \rangle$, where $X_1 = X$, $X_{k+1} = Y$, and all the vertices $X_1, X_2, \ldots, X_{k+1}$ are distinct except possibly $X_1 = X_{k+1}$. For convenience, we use the sequence $\langle X_1, \ldots, X_i, P(X_i, X_j), X_j, \ldots, X_{k+1} \rangle$ to denote the path $\langle X_1, X_2, \ldots, X_{k+1} \rangle$, where $P(X_i, X_j) = \langle X_i, X_{i+1}, \ldots, X_j \rangle$ and the two vertices X_i and X_j are called the *end-vertices* of $P(X_i, X_j)$. We call that $P(X_i, X_j)$ is a *sub-path* of the path from X to Y. Sometimes, we also use P to denote a path $P(X_i, X_j)$. Let $l(P(X_i, X_j))$ denote the length of the path P that is the number of edges in P. The *distance* between X and Y in G is denoted by $d_G(X, Y)$, which is the length of a shortest path between X and Y in G. A *cycle* C is a special path with at least three vertices such that the first vertex is the same as the last one. A cycle of length k is called a k-cycle. A path (respectively, cycle) which traverses each vertex of G exactly once is *hamiltonian path* (respectively, *hamiltonian cycle*).

The n-dimensional Möbius cube MQ_n, proposed first by Cull and Larson [2], consists of 2^n vertices and each vertex has a unique n-component binary vector for an address. Each vertex has n neighbors as follows. A vertex $X = x_1 x_2 \ldots x_n$ connects to its ith neighbor, denoted by $N_i(X)$, for $2 \leq i \leq n$, $N_i(X) = x_1 x_2 \ldots x_{i-1} \overline{x}_i x_{i+1} \ldots x_n$ if $x_{i-1} = 0$ or $N_i(X) = x_1 x_2 \ldots x_{i-1} \overline{x}_i \overline{x}_{i+1} \ldots \overline{x}_n$ if $x_{i-1} = 1$.

For $i = 1$, since there is no bit on the left of x_1, $N_1(X)$ can be defined as the first neighbor of

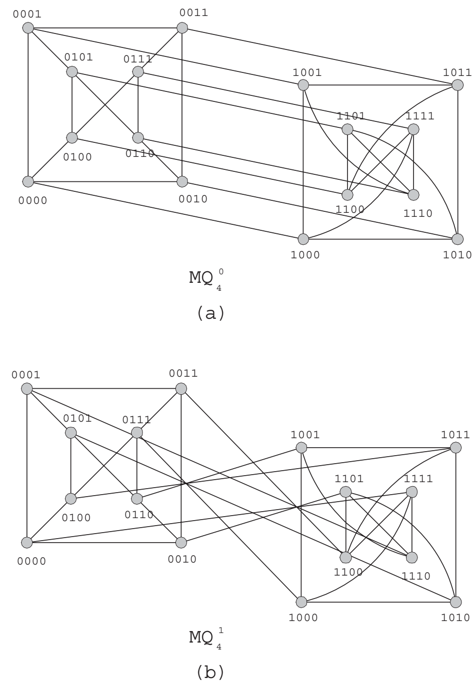

Figure 1. (a) A 0-type 4-dimensional Möbius cube. (b) A 1-type 4-dimensional Möbius cube.

X can be denoted as $\overline{x}_1 x_2 \ldots x_n$ or $\overline{x}_1 \overline{x}_2 \ldots \overline{x}_n$. If we assume that the zeroth bit of every vertex of MQ_n is 0, we call the network a *0-type* n-dimensional Möbius cube, denoted by MQ_n^0; and if we assume that the zeroth bit of every vertex of MQ_n is 1, we call the network a *1-type* n-dimensional Möbius cube, denoted by MQ_n^1. Either MQ_n^0 or MQ_n^1 may be denoted by MQ_n. The example of MQ_4^0 and MQ_4^1 are shown in Fig 1.

Therefore, MQ_n is an n-regular graph and can be recursively defined as follows: Both MQ_1^0 and MQ_1^1 are complete graph K_2 with one vertex labeled 0 and the other 1. MQ_n^0 and MQ_n^1 are both composed of a sub-Möbius cube MQ_{n-1}^0 and a sub-Möbius cube MQ_{n-1}^1. Each vertex

$X = 0x_2x_3 \dots x_{n-1}x_n \in V(MQ^0_{n-1})$ connects to $1x_2x_3 \dots x_{n-1}x_n \in MQ^1_{n-1}$ in MQ^0_n and to $1\overline{x}_2\overline{x}_3 \dots \overline{x}_{n-1}\overline{x}_n$ in MQ^1_n. For convenience, we say that MQ^0_{n-1} and MQ^1_{n-1} are two *sub-Möbius cubes* of MQ_n, where MQ^0_{n-1} (respectively, MQ^1_{n-1}) is an $(n-1)$-dimensional 0-type Möbius cube (respectively, 1-type Möbius cube) which includes all vetices $0x_2x_3 \dots x_{n-1}x_n$ (respectively, $1x_2x_3 \dots x_{n-1}x_n$), $x_i \in \{0, 1\}$.

Let e_i be the n-dimensional $(0, 1)$ vector with only its ith component equal to 1. Let E_i be the n-dimensional $(0, 1)$ vector with ith through nth components equal to 1. Let $\{Z_2\}^n$ be the n-dimensional vector space over $\{0, 1\}$ with addition and scalar multiplication mod 2. It is clear that both $\{e_i \mid 1 \leq 1 \leq n\}$ and $\{E_i \mid 1 \leq i \leq n\}$ are bases for this space. Hence $\{e_i, E_i \mid 1 \leq i \leq n\}$ forms a redundant basis for this vector space. Any vector X can be represented as a linear sum of these basis vectors:

$$X = \sum_{i=1}^{n} (\alpha_i e_i + \beta_i E_i), \qquad (1)$$

where $\alpha_i \in \{0, 1\}$ and $\beta_i \in \{0, 1\}$. Clearly, we can represent a vector X by the set of vectors e_i, E_i that have nonzero coefficients in the above sum. For any vertex X in MQ_n, the ith neighbor of X, $N_i(X)$, is formed by $X + e_i$ if $x_{i-1} = 0$, or $X + E_i$ if $x_{i-1} = 1$. It is clearly that every vertex X of MQ_n can be formulated as the above sum.

Definition 1 *[2] A set S of e_i, E_i, where $1 \leq i \leq n$, is an expansion of X if and only if the equality in* (1) *is true, where $\alpha_i = 1$ if and only if $e_i \in S$ and $\beta_i = 1$ if and only if $E_i \in S$. Also, any $t \in S$ is called a term of this expansion of X.*

Because we are using a redundant basis, there can be more than one expansion of a vector. For a vector X, the weight of an expansion S of X is the cardinality of set S, denoted by $|S|$ and a minimal expansion of X is an expansion with least weight.

Lemma 1 *Let X be a vertex of MQ^0_n with $n \geq 3$ and $Y = N_i(X)$. Then $d_{MQ^0_n}(N_1(X), N_1(Y)) = 1$ if $3 \leq i \leq n$ and $d_{MQ^0_n}(N_1(X), N_1(Y)) = 2$ if $i = 2$.*

Proof. Let $X = x_1x_2 \dots x_{i-1}x_ix_{i+1} \dots x_n$ where $x_j \in \{0, 1\}$ for $1 \leq j \leq n$. Since Y is an ith neighbor of X, $Y = x_1x_2 \dots x_{i-1}\overline{x}_ix_{i+1} \dots x_n$ if $x_{i-1} = 0$ or $Y = x_1x_2 \dots x_{i-1}\overline{x}_i\overline{x}_{i+1} \dots \overline{x}_n$ if $x_{i-1} = 1$.

Case 1: $i = 2$.

Suppose that $x_1 = 0$. Then, $N_1(X) = 1x_2x_3 \dots x_n$ and $N_1(Y) = 1\overline{x}_2x_3 \dots x_n$. By definition, $d_{MQ^0_n}(N_1(X), N_1(Y)) > 1$. If $x_2 = 0$, $N_1(Y) + E_3 = 1\overline{x}_2\ \overline{x}_3 \dots \overline{x}_n$. Hence $N_1(Y) + E_3 + E_2 = 1x_2x_3 \dots x_n$. Hence $d_{MQ^0_n}(N_1(X), N_1(Y)) = 2$. If $x_2 = 1$, $N_1(X) + E_3 = 1x_2\overline{x}_3 \dots \overline{x}_n$. Hence $N_2(X) + E_3 + E_2 = 1\overline{x}_2x_3 \dots x_n$. Therefore, $d_{MQ^0_n}(N_1(X), N_1(Y)) = 2$.

Suppose that $x_1 = 1$. Then, $N_1(X) = 0x_2x_3 \dots x_n$ and $N_1(Y) = 0\overline{x}_2\overline{x}_3 \dots \overline{x}_n$. By definition, $d_{MQ^0_n}(N_1(X), N_1(Y)) > 1$. If

$x_2 = 0$, $N_1(Y) + E_3 = 0\overline{x}_2x_3\ldots x_n$. Hence $N_1(Y) + E_3 + e_2 = 0x_2x_3\ldots x_n$. Hence $d_{MQ_n^0}(N_1(X), N_1(Y)) = 2$. If $x_2 = 1$, $N_1(X) + E_3 = 0x_2\overline{x}_3\ldots\overline{x}_n$. Hence $N_1(X) + E_3 + e_2 = 0\overline{x}_2\overline{x}_3\ldots\overline{x}_n$. Therefore, $d_{MQ_n^0}(N_1(X), N_1(Y)) = 2$.

Case 2: $3 \le i \le n$.

Suppose that $x_{i-1} = 0$. $N_1(X) = \overline{x}_1x_2\ldots x_{i-2}0x_i\ldots x_n$ and $N_1(Y) = \overline{x}_1x_2\ldots x_{i-2}0\overline{x}_ix_{i+1}\ldots x_n$. It is obvious that $N_1(Y) + e_i = N_1(X)$. Hence $d_{MQ_n^0}(N_1(X), N_1(Y)) = 1$.

Suppose that $x_{i-1} = 1$. $N_1(X) = \overline{x}_1x_2\ldots x_{i-2}1x_i\ldots x_n$ and $N_1(Y) = \overline{x}_1x_2\ldots x_{i-2}1\overline{x}_i\ldots\overline{x}_n$. It is obvious that $N_1(Y) + E_i = N_1(X)$. Hence $d_{MQ_n^0}(N_1(X), N_1(Y)) = 1$. The lemma is proved.

Lemma 2 *Let X be a vertex of MQ_n^1 with $n \ge 3$ and $Y = N_i(X)$. Then $d_{MQ_n^1}(N_1(X), N_1(Y)) = 1$ if $i = n$ and $d_{MQ_n^1}(N_1(X), N_1(Y)) = 2$ if $2 \le i \le n-1$.*

Proof. Let $X = x_1x_2\ldots x_{i-1}x_ix_{i+1}\ldots x_n$ where $x_j \in \{0,1\}$ for $1 \le j \le n$. Since Y is an ith neighbor of X, $Y = x_1x_2\ldots x_{i-1}\overline{x}_ix_{i+1}\ldots x_n$ if $x_{i-1} = 0$ or $Y = x_1x_2\ldots x_{i-1}\overline{x}_i\overline{x}_{i+1}\ldots\overline{x}_n$ if $x_{i-1} = 1$.

Case 1: $2 \le i \le n-1$.

Suppose that $x_{i-1} = 0$. $N_1(X) = \overline{x}_1\overline{x}_2\ldots\overline{x}_{i-2}1\overline{x}_i\ldots\overline{x}_n$ and $N_1(Y) = \overline{x}_1\overline{x}_2\ldots\overline{x}_{i-2}1x_i\overline{x}_{i+1}\ldots\overline{x}_n$. By definition, $d_{MQ_n^0}(N_1(X), N_1(Y)) > 1$. If $x_i = 0$, $N_1(X) + E_{i+1} = \overline{x}_1\overline{x}_2\ldots\overline{x}_{i-2}1\overline{x}_ix_{i+1}\ldots x_n$. Hence $N_1(X) + E_{i+1} + E_i = \overline{x}_1\overline{x}_2\ldots\overline{x}_{i-2}1x_i\overline{x}_{i+1}\ldots\overline{x}_n$. Hence $d_{MQ_n^0}(N_1(X), N_1(Y)) = 2$. If $x_i = 1$, $N_1(Y) + E_{i+1} = \overline{x}_1\overline{x}_2\overline{x}_3\ldots\overline{x}_{i-2}1x_ix_{i+1}\ldots x_n$. Hence $N_1(Y) + E_{i+1} + E_i = N_1(X)$. Therefore, $d_{MQ_n^0}(N_1(X), N_1(Y)) = 2$.

Suppose that $x_{i-1} = 1$. $N_1(X) = \overline{x}_1\overline{x}_2\ldots\overline{x}_{i-2}0\overline{x}_i\ldots\overline{x}_n$ and $N_1(Y) = \overline{x}_1\overline{x}_2\ldots\overline{x}_{i-2}0x_ix_{i+1}\ldots x_n$. By definition, $d_{MQ_n^0}(N_1(X), N_1(Y)) > 1$. If $x_i = 0$, $N_1(X) + E_{i+1} = \overline{x}_1\overline{x}_2\ldots\overline{x}_{i-2}0\overline{x}_ix_{i+1}\ldots x_n$. Hence $N_1(X) + E_{i+1} + e_i = N_1(Y)$. Hence $d_{MQ_n^0}(N_1(X), N_1(Y)) = 2$. If $x_i = 1$, $N_1(Y) + E_{i+1} = \overline{x}_1\overline{x}_2\ldots\overline{x}_{i-2}0x_i\overline{x}_{i+1}\ldots\overline{x}_n$. Hence $N_1(Y) + E_{i+1} + e_i = N_1(X)$. Therefore, $d_{MQ_n^0}(N_1(X), N_1(Y)) = 2$.

Case 2: $i = n$.

$N_1(X) = \overline{x}_1\overline{x}_2\ldots\overline{x}_{n-2}\overline{x}_{n-1}\overline{x}_n$ and $N_1(Y) = \overline{x}_1\overline{x}_2\ldots\overline{x}_{n-2}\overline{x}_{n-1}x_n$. It is obvious that $N_1(Y) + e_n = N_1(X)$. Hence $d_{MQ_n^0}(N_1(X), N_1(Y)) = 1$.

Cull and Larson [2] proposed an algorithm to generate a minimal expansion of a vector of X using only components x_i through x_n as $S(X, i)$ for $1 \le i \le n$.

Algorithm $S(X, i)$

Input: *A vector X and an integer i with $1 \le i \le n$.*

Output: *A minimal expansion of X using components x_i through x_n.*

begin

if $X = ()$ *then return empty set.*

if $X = (1)$ *then return* $\{E_i\}$.

if $X = (0X')$ *then return* $S(X', i + 1)$.

if $X = (10X')$ *then return* $\{e_i\} \cup S(X', i + 2)$.

if $X = (11X')$ *then return* $\{E_i\} \cup S(\overline{X'}, i + 2)$.

end

Lemma 3 *[2] The "greedy" algorithm given in correctly produces a minimal expansion of* X*, by computing* $S(X, 1)$.

Lemma 4 *[2] Let* X *and* Y *be two vertices of* MQ_n*, and* S *be a minimal expansion of* $X + Y$ *produced by the "greedy" minimal expansion algorithm. Then* $d_{MQ_n}(X, Y) = |S|$ *or* $|S| + 1$.

Lemma 5 *Let* X *and* Y *be two distinct vertices in* MQ_n*. Then* $d_{MQ_n}(X, Y) = d_{MQ_n}(N_1(X), N_1(Y)) \pm k$ *where* $k = 0, 1$.

Proof. Let $X = x_1 x_2 \ldots x_n$ and $Y = y_1 y_2 \ldots y_n$. Also let S be a minimal expansion of $X + Y$ produced by the "greedy" minimal expansion algorithm. It is observed that e_1 and E_1 doesn't be contained in S. Suppose that X and Y are in MQ^0_n. Hence $N_1(X) = \overline{x}_1 x_2 \ldots x_n$ and $N_1(Y) = \overline{y}_1 y_2 \ldots y_n$. Suppose that X and Y are in MQ^1_n. First neighbors of X and Y are $N_1(X) = \overline{x}_1 \overline{x}_2 \ldots \overline{x}_n$ and $N_1(Y) = \overline{y}_1 \overline{y}_2 \ldots \overline{y}_n$, respectively. One may observe that $X + Y = N_1(X) + N_1(Y)$. Consequently, S is a minimal expansion of $N_1(X) + N_1(Y)$. By Lemma 4, we have that $d_{MQ_n}(X, Y) = |S|$ or $d_{MQ_n}(X, Y) = |S| + 1$, and $d_{MQ_n}(N_1(X), N_1(Y)) = |S|$ or $d_{MQ_n}(N_1(X), N_1(Y)) = |S| + 1$. Therefore, $d_{MQ_n}(X, Y) = d_{MQ_n}(N_1(X), N_1(Y)) \pm k$ where $k = 0, 1$.

Lemma 6 *Let* X *and* Y *be two vertices in the same sub-Möbius* MQ^i_{n-1} *of* MQ_n *with* $i = 0, 1$*. Then every shortest path* $P_s(X, Y)$ *joining* X *and* Y *in* MQ_n *satisfies that all vertices on* $P_s(X, Y)$ *belong to* MQ^i_{n-1}.

Proof. Without loss of generality, we assume that X and Y are two vertices in MQ^0_{n-1}. Let $P_s(X, Y)$ be a shortest path joining X and Y in MQ_n. Suppose that there exists a sub-path of $P_s(X, Y)$ in MQ^1_{n-1}. Let $P_s(X, Y)$ is formed by $\langle X, P_s(X, U), U, W, P_s(W, Z), Z, S, P_s(S, Y), Y \rangle$ where $W = N_1(U)$, $Z = N_1(S)$, and $P_s(W, Z)$ lies on MQ^1_{n-1}. Hence $U, S \in V(MQ^0_{n-1})$. Since the path $P_s(X, Y)$ is a shortest path joining X and Y in MQ_n, the path $\langle U, W, P_s(W, Z), Z, S \rangle$ is a shortest path between U, S in MQ_n. Therefore, $d_{MQ_n}(U, S) = d_{MQ_n}(W, Z) + 2$. Since $W = N_1(U)$, $Z = N_1(S)$, and by Lemma 5, $d_{MQ_n}(U, S) = d_{MQ_n}(W, Z) \pm k$ where $k = 0, 1$. This is contradiction. Consequently, there is no sub-path of $P_s(X, Y)$ in MQ^1_{n-1}. The lemma follows.

Lemma 7 *Let* $X \in V(MQ^i_{n-1})$ *and* $Y \in V(MQ^{1-i}_{n-1})$ *be two vertices in* MQ_n*. Then there* *exists a shortest path $P_s(X,Y)$ joining X and Y forms of $\langle X$, $N_1(X)$, $P_s(N_1(X),Y)$, $Y\rangle$ or $\langle X$, $P_s(X,N_1(Y))$, $N_1(Y)$, $Y\rangle$ where $P_s(X,N_1(Y))$ in MQ^i_{n-1} and $P_s(N_1(X),Y)$ in MQ^{1-i}_{n-1}.*

Proof. Let S be a minimal expansion of $X+Y$ produced by the "greedy" minimal expansion algorithm. Assume that the lowest index term in S has index i. It is clearly that routing along any edge $(X,X+t_j)$, $j>i$ doesn't affect bit x_i and routing along any edge $(X,X+t_j)$, $j<i$ doesn't lead to minimal path from X to Y where $t_j\in S$. So the shortest path algorithm must eventually rout along only one of the edges $(Z,Z+e_i)$ or $(Z,Z+E_i)$ for some vertex Z on the path between X and Y. Since $X\in V(MQ^i_{n-1})$ and $Y\in V(MQ^{1-i}_{n-1})$, $X=x_1x_2x_3\dots x_n$ and $Y=\overline{x}_1y_2y_3\dots y_n$. Hence the lowest index term of S has index 1. Therefore, an exact minimal routing algorithm given in [2] can determine a shortest path $P_s(X,Y)$ between X and Y such that $P_s(X,Y)$ forms of $\langle X,N_1(X),P_s(N_1(X),Y),Y\rangle$ or $\langle X,P_s(X,N_1(Y)),Y\rangle$ where $P_s(X,N_1(Y))$ in MQ^i_{n-1} and $P_s(N_1(X),Y)$ in MQ^{1-i}_{n-1}. The lemma follows.

A graph G is *panconnected* if each pair of distinct vertices X and Y are joined by a path of length l where $d_G(X,Y)\le l\le |V(G)|-1$. The following panconnected property of MQ_n are useful in the proof of next section.

Lemma 8 *[11] If $n\ge 3$ then for any two distinct vertices X and Y in MQ_n, there exists a path of every length from $d_{MQ_n}(X,Y)+2$ to 2^n-1.*

The diameter $D(G)$ of G is the maximal value of distances between all pairs of vertices in G. It is clearly that $D(MQ_3)=2$.

Lemma 9 *[2] The diameter of the n-dimensional Möbius cube MQ_n is $D(MQ^0_n)=\lceil\frac{n+2}{2}\rceil$ for $n\ge 4$ and $D(MQ^1_n)=\lceil\frac{n+1}{2}\rceil$ for $n\ge 1$.*

3 MQ_n is geodesic 3-pancyclic

Definition 2 *Let G be a graph. For two vertices $X,Y\in V(G)$, a cycle is called a geodesic cycle with X and Y if a shortest path joining X and Y lies on the cycle. A geodesic l-cycle with X and Y in G, denoted by $gC^l(X,Y;G)$, is a geodesic cycle of length l.*

Definition 3 *Let G be a graph. For two vertices $X,Y\in V(G)$, they are called geodesic k-pancyclic on X and Y if for every integer l satisfying $2d_G(X,Y)+k\le l\le |V(G)|$, the geodesic cycle $gC^l(X,Y;G)$ exists.*

Definition 4 *The graph G is called geodesic k-pancyclic if any distinct two vertices on G are geodesic k-pancyclic on them. The geodesic-pancyclicity of G, denoted by $gpc(G)$, is defined as the minimum integer k such that G is geodesic k-pancyclic.*

This section is dedicated to illustrating the geodesic pancyclic property of Möbius cubes. We first propose that MQ_3 is geodesic 2-pancyclic. Finally, we prove that $2 \leq gpc(MQ_n) \leq 3$ for $n \geq 4$.

Lemma 10 *MQ_3 is geodesic 2-pancyclic.*

Proof. Since MQ_3^0 and MQ_3^1 are isomorphic, we only prove the case of MQ_3^0. Since MQ_3 is vertex-transitive. We assume that $X = 000$ and consider Y as the four cases: (1) $Y \in \{100, 010\}$, (2) $Y = 001$, (3) $Y = \{110, 111\}$, and (4) $Y \in \{101, 011\}$. By the symmetry of MQ_3, there is only one vertex is discussed for each case and related geodesic cycles are listed as Table 1.

Theorem 1 *MQ_n is geodesic 3-pancyclic for $n \geq 3$.*

Proof. The theorem is proved by induction on n. By Lemma 10, MQ_3 is geodesic 2-pancyclic. This implies that MQ_3 is geodesic 3-pancyclic. The theorem holds for $n = 3$. Assume that the theorem is true for every integer $3 \leq m < n$. We now consider $m = n$ as follows. Let X and Y be two vertices in MQ_n. By the relative position of X and Y, the proof is divided into two parts: **(1)** X and Y are in the same sub-Möbius MQ_{n-1}^i and **(2)** $X \in V(MQ_{n-1}^i)$ and $Y \in V(MQ_{n-1}^{1-i})$ for $i = 0, 1$.

Case 1: $X, Y \in V(MQ_{n-1}^i)$ for $i = 0, 1$.

Let $d_{MQ_n}(X, Y) = d$. Without loss of generality, we may assume that $X, Y \in V(MQ_{n-1}^0)$. By the induction hypothesis, we

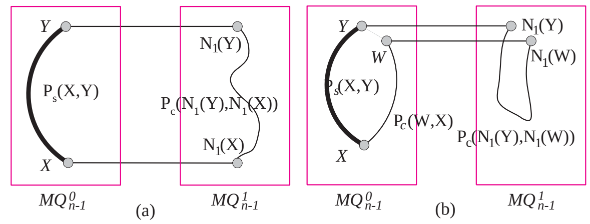

Figure 2. Two examples for case 1 of Theorem 1

have the geodesic cycle $gC^l(X, Y; MQ_{n-1}^0)$ for all $2d_{MQ_{n-1}^0}(X, Y) + 3 \leq l \leq 2^{n-1}$. By Lemma 6, $d_{MQ_n}(X, Y) = d_{MQ_{n-1}^0}(X, Y) = d$. Therefore, the geodesic cycle $gC^l(X, Y; MQ_n)$ for all $2d + 3 \leq l \leq 2^{n-1}$ follows.

We now construct the geodesic cycle $gC^l(X, Y; MQ_n)$ for all $2^{n-1} + 1 \leq l \leq 2^{n-1}$. By Lemma 5, $d_{MQ_{n-1}^1}(N_1(X), N_1(Y)) = d_{MQ_n}(N_1(X), N_1(Y)) = d$ or $d + 1$. By Lemma 8, there exists a path of $\langle N_1(Y)$ $, P_c(N_1(Y), N_1(X)), N_1(X)\rangle$ in MQ_{n-1}^1 where $d + 3 \leq l(P_c(N_1(Y), N_1(X)) \leq 2^{n-1} - 1$. Let cycle $C = \langle X, P_s(X, Y), Y, N_1(Y), P_c(N_1(Y), N_1(X)), N_1(X), X\rangle$. Then, $2d + 5 \leq l(C) \leq 2^{n-1} + d + 1$. Since $d \leq D(MQ_{n-1})$, $2d + 5 \leq 2^{n-1} + 1$ for $n \geq 4$. Therefore, the geodesic cycle $gC^l(X, Y; MQ_n)$ exists where $2^{n-1} + 1 \leq l \leq 2^{n-1} + d + 1$. (See Figure 2 (a).)

It is difficult to prove that the geodesic cycle $gC^5(X, Y : MQ_3)$ exists for any X, Y in MQ_3. Since $2^n - 3 > 2 \times D(MQ_n) + 3$ for $n \geq 4$, there exists the geodesic cycle $gC^{2^n - 3}(X, Y; MQ_n)$ on any two distinct vertices X and Y in MQ_n for $n \geq 3$. Let $gC^{2^{n-1}-3}(X, Y; MQ_{n-1}^0)$=$\langle X$, $P_s(X, Y)$, Y,

Table 1. Summary of the geodesic cycles with $X = 000$ and Y in MQ_3^0.

Y	geodesic cyclic (even length)	geodesic cyclic (odd length)
100	$\langle 000, \underline{100}, 101, 001, 000\rangle$	$\langle 000, \underline{100}, 111, 011, 010, 000\rangle$
100	$\langle 000, \underline{100}, 111, 110, 101, 001, 000\rangle$	$\langle 000, \underline{100}, 101, 110, 111, 011, 001, 000\rangle$
100	$\langle 000, \underline{100}, 111, 011, 001, 101, 110, 010, 000\rangle$	
001	$\langle 000, \underline{001}, 011, 010, 000\rangle$	$\langle 000, \underline{001}, 011, 111, 100, 000\rangle$
001	$\langle 000, \underline{001}, 011, 111, 110, 010, 000\rangle$	$\langle 000, \underline{001}, 011, 111, 110, 101, 100, 000\rangle$
001	$\langle 000, \underline{001}, 011, 111, 100, 101, 110, 010, 000\rangle$	
110		$\langle 000, 010, \underline{110}, 111, 100, 000\rangle$
110	$\langle 000, 010, \underline{110}, 111, 011, 001, 000\rangle$	$\langle 000, 010, \underline{110}, 111, 100, 101, 001, 000\rangle$
110	$\langle 000, 010, \underline{110}, 101, 001, 011, 111, 100, 000\rangle$	
011	$\langle 000, 001, \underline{011}, 010, 000\rangle$	$\langle 000, 001, \underline{011}, 111, 100, 000\rangle$
011	$\langle 000, 001, \underline{011}, 111, 110, 010, 000\rangle$	$\langle 000, 001, \underline{011}, 111, 110, 101, 100, 000\rangle$
011	$\langle 000, 001, \underline{011}, 111, 100, 101, 110, 010, 000\rangle$	

$P_c(Y, X)$, $X\rangle$ where $l(P_s(X, Y)) = d$. Let W and Y be two adjacent vertices on $P_c(Y, X)$. Hence $P_c(Y, X) = \langle Y, W, P_c(W, X), X\rangle$ where $W = N_j(Y)$ for some j. Since $W = N_j(Y)$ for some $2 \leq j \leq n$ and by Lemma 5, $d_{MQ_n}(N_1(Y), N_1(W)) \leq 2$. By Lemma 8, there exists a path of $\langle N_1(Y)$, $P_c(N_1(Y), N_1(W))$, $N_1(W)\rangle$ in MQ_{n-1}^1 where $4 \leq l(P_c(N_1(Y), N_1(W))) \leq 2^{n-1} - 1$. Let cycle $C = \langle X$, $P_s(X, Y)$, $Y, N_1(Y)$, $P_c(N_1(Y), N_1(W))$, $N_1(W), W, P_c(W, X), X\rangle$. Then $2^{n-1} + 3 \leq l(C) \leq 2^n - 3$. Since $d \geq 1$, we have the geodesic cycle $gC^l(X, Y; MQ_n)$ for all $2^{n-1} + d + 2 \leq l \leq 2^n - 3$ with format C. (See Figure 2 (b).) Similarly, the geodesic cycle $gC^{2^n}(X, Y; MQ_n)$ for all $2^n - 2 \leq l \leq 2^n$ may be obtained if the geodesic cycle $gC^{2^{n-1}}(X, Y; MQ_{n-1}^0)$ is used in the construction method. Hence, this case holds.

Case 2: $X \in V(MQ_{n-1}^i)$ and $Y \in V(MQ_{n-1}^{1-i})$ for $i = 0, 1$.

Without loss of generality, let $X \in V(MQ_{n-1}^0)$ and $Y \in V(MQ_{n-1}^1)$. According to relationship of X and Y, the proof of this case is divided into two parts: (1) $Y = N_1(X)$, i.e., X and Y are adjacent. (2) $Y \neq N_1(X)$, i.e., X and Y are not adjacent.

Subcase 2.1 $Y = N_1(X)$.

By Lemma 1-2, $N_n(Y)$ and $N_n(X)$ are adjacent. By Lemma 8, any path of $\langle N_n(Y)$, $P_c(N_n(Y), Y), Y\rangle$ exists in MQ_{n-1}^1 where $1, 3 \leq l(P_c(N_n(Y), Y) \leq 2^{n-1} - 1$ and there exists a path of $\langle X$, $P_c(X, N_n(X))$, $N_n(X)\rangle$ in MQ_{n-1}^0 where $1, 3 \leq l(P_c(X, N_n(X)) \leq 2^{n-1} - 1$. Let cycle $C = \langle X$, $P_c(X, N_n(X))$, $N_n(X)$, $N_n(Y)$, $P_c(N_n(Y), Y)$, $Y, X\rangle$. Then $4, 6 \leq l(C) \leq 2^n$. By Lemma 1-2, $d_{MQ_n}(Y, N_1(N_2(X)) = 2$, the geodesic cycle $gC^5(X, Y; MQ_n)$ can be found. Hence, we have the geodesic cycle $gC^l(X, Y; MQ_n)$ for all $4 \leq l \leq 2^n$.

Subcase 2.2 $Y \neq N_1(X)$.

By Lemma 7, without loss of general-

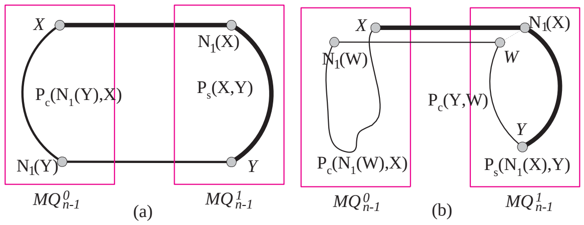

Figure 3. Two examples for subcase 2.2 of Theorem 1

ity, we may assume there exists a shortest path $P_s(X, Y)$ with format of $\langle X,\ N_1(X),\ P_s(N_1(X), Y),\ Y\rangle$ where $P_s(N_1(X), Y)$ is a shortest path joining $N_1(X)$ and Y in MQ^1_{n-1} (See Figure **??** (a)). Let $d_{MQ_n}(X, Y) = d$. Hence $d_{MQ_n}(N_1(X), Y) = d - 1$. By Lemma 5, we have that $d_{MQ_n}(X, N_1(Y)) = d - 1$ or d. By Lemma 8, there exists a path $P_c(N_1(Y), X)$ in MQ^0_{n-1} where $d + 2 \leq l(P_c(N_1(Y), X)) \leq 2^{n-1} - 1$. Let cycle $C = \langle X,\ P_s(X, Y),\ Y,\ N_1(Y),\ P_c(N_1(Y), X),\ X\rangle$. Then, $2d + 3 \leq l(C) \leq 2^{n-1} + d$. Consequently, there exists the geodesic cycle $gC^l(X, Y; MQ_n)$ for all $2d+3 \leq l \leq 2^{n-1} + d$ with format C.

We now construct the geodesic cycle $gC^l(X, Y; MQ_n)$ for all $2^{n-1} + d + 1 \leq l \leq 2^n$ (See Figure **??** (b)). By the induction hypothesis, the geodesic cycle $gC^l(N_1(X), Y; MQ^1_{n-1})$ for all $2(d - 1) + 3 \leq l \leq 2^{n-1}$ exits. It is observed that for any two distinct vertices A, B in MQ_n with $n \geq 3$, the cycle $gC^{2^n-2}(A, B; MQ_n)$ exists. Let $gC^{2^{n-1}-2}(N_1(X), Y; MQ^1_{n-1}) = \langle N_1(X),\ P_s(N_1(X), Y),\ Y,\ P_c(Y, N_1(X)),\ N_1(X)\rangle$. Let W be the adjacent vertex of $N_1(X)$ on $P_c(Y, N_1(X))$. Hence $P_c(Y, N_1(X)) = \langle Y,\ P_c(Y, W),\ W,\ N_1(X)\rangle$. By Lemma 1-2, $d_{MQ_n}(X, N_1(W)) \leq 2$. By Lemma 8, there exists a path $P_c(N_1(W), X)$ in MQ^0_{n-1} where $4 \leq l(P_c) \leq 2^{n-1} - 1$. Let cycle $C = \langle X,\ N_1(X), P_s(N_1(X), Y),\ Y, P_c(Y, W),\ W, N_1(W),\ P_c(N_1(W), X), X\rangle$. Then the length of cycle C is $l(P_s(N_1(X), Y)) + l(P_c(Y, W)) + l(P_c(N_1(W), X)) + 2$. It is obvious that $2^{n-1} + 3 \leq l(C) \leq 2^n - 2$. Since $d \geq 2$, there exists the geodesic cycle $gC^l(X, Y; MQ_n)$ for all $2^{n-1} + d + 1 \leq l \leq 2^n - 2$. Similarly, the geodesic cycle $gC^{2^n-1}(X, Y; MQ_n)$ and $gC^{2^n}(X, Y; MQ_n)$ may be obtained if the geodesic cycle $gC^{2^{n-1}}(N_1(X), Y; MQ^1_{n-1})$ is used in this construction method. Hence, this case holds.

It is well known that there is no triangle cycle in MQ_n. Therefore, there is no geodesic 1-cycle with two adjacent vertices in MQ_n. Hence $gpc(MQ_n) \geq 2$. Then the following corollary holds.

Corollary 1 *The geodesic-pancyclicity of MQ_n is $gpc(MQ_3) = 2$ and $2 \leq gpc(MQ_n) \leq 3$ for $n \geq 4$.*

4 Conclusions

In this paper, we demonstrate that for any two vertices X and Y in MQ_n for $n \geq 3$, there exists a geodesic l-cycle on them where $2d_{MQ_n}(X, Y) + 3 \leq l \leq 2^n$. We show that $2 \leq gpc(MQ_n) \leq 3$ for $n \geq 3$. This result is near optimal because there is no geodesic 1-cycle with two adjacent vertices in MQ_n. We have a conjecture that $gpc(MQ_n) = 2$ because not all pair of vertices X and Y does not exist a path of length $d_{MQ_n}(X,Y)+1$ between them.

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