(n, k)-星形圖之弱節點泛迴圈性質

Weak-Vertex-Pancyclicity of (n, k)-Star Graphs

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摘要

(n, k)-星狀圖（簡稱 $S_{n,k}$）是一個超立方體的極佳替代圖，也是 n-星狀圖的一般化版本。$S_{n,n-1}$ 與 n-星狀圖是同構的，且 $S_{n,1}$ 亦與 n-完全圖同構。n-星狀圖已經被證明包含所有長度從 6 到 $n!$的偶數迴圈。而本研究則證明了當 $1 \le k \le n-4$ 及 $n \ge 6$ 時，$S_{n,k}$ 的任一節點包含於長度在 3 到 $|V(S_{n,k})|$之間的迴圈。另外，當 $n-3 \le k \le n-2$ 時，$S_{n,k}$ 的任一節點包含於長度為 6 到 $|V(S_{n,k})|$的迴圈。此外，在 $S_{n,k}$ 中所建構的每一個迴圈皆可包含一指定邊，且該邊包含於某一 $S_{n-k+1,1}$ 子圖中。

關鍵詞：弱節點泛迴圈，內含迴圈，(n, k)-星形圖，n-星形圖，互連網路

Abstract

The (n, k)-star graph ($S_{n,k}$ for short) is an attractive alternative to the hypercube and also a generalized version of the n-star. It is isomorphic to the n-star (n-complete) graph if $k = n-1$ ($k = 1$). Jwo *et al.* have already demonstrated in 1991 that an n-star contains a cycle of every even length from 6 to $n!$. This work shows that every vertex in an $S_{n,k}$ lies on a cycle of length l for every $3 \le l \le n!/(n-k)!$ when $1 \le k \le n-4$ and $n \ge 6$. Additionally, for $n-3 \le k \le n-2$, each vertex in an $S_{n,k}$ is contained in a cycle of length ranged from 6 to $n!/(n-k)!$. Moreover, each constructed cycle of an available length in an $S_{n,k}$ can contain a desired 1-edge.

Keywords: Weak-vertex-pancyclicity, cycle embedding, (n, k)-star graph, n-star graph, interconnection networks

1 Introduction

The n-star graph (S_n for short), which proposed by Akers *et al.* in 1987, is an attractive alternative to the hypercube with vertex and edge symmetry [1]. An S_n is a Cayley graph with a regular and hierarchical structure; for a similar number of vertices, the graph has a lower vertex degree, a smaller diameter, and a shorter average distance than the comparable hypercube. However, with the restriction on the number of vertices: $n!$, there is a large gap between $n!$ and $(n+1)!$ for expanding an S_n to an S_{n+1}.

To relax the restriction of the numbers

of vertices $n!$ in an S_n, a generalized version of the star graph, the (n, k)-star graph, was proposed in 1995 [4]. An $S_{n,k}$ preserves many attractive properties of an S_n such as vertex symmetry, hierarchical structure, maximal fault tolerance, and simple shortest routing. The two parameters n and k can be tuned to make a suitable choice for the number of vertices in the network and for the degree/diameter tradeoff. An $S_{n,k}$ is regular of degree n–1, and the number of vertices is $n!/(n–k)!$.

Some basic properties of an $S_{n,k}$, such as diameter [4], connectivity [4], broadcasting [7], average distance [5], embedding [6], Hamiltonicity [8], spanning connectivity [9], and wide diameter [11], have recently been computed or derived. These measurement results demonstrate that an $S_{n,k}$ has excellent topological properties.

Path and cycle are two of the most fundamental networks for parallel and distributed computation, and suitable for designing simple algorithms with low communication costs. The *pancyclicity* of a network represents its power of embedding cycles of all possible lengths. A graph G of order $|V(G)|$ is called *pancyclic* whenever G contains a cycle of each length l for $3 \le l \le |V(G)|$ [3]. Various generalizations of pancyclic graphs have been studied, such as Hypercube and Pyramid graphs. A graph G is *bipancyclic* if it has a cycle of every even length from 4 to $|V(G)|$. A graph G is *vertex-pancyclic* (*vertex-bipancyclic*) if every vertex lies on a cycle of every length (every even length) from 3 (4) to $|V(G)|$. An *m-weak-pancyclic* (*m-weak-bipancyclic*) graph is a graph which contains a cycle of every length (every even length) l for $m \le l \le |V(G)|$. Additionally, a graph G is *m-weak-vertex-pancyclic* (*m-weak-vertex-bipancyclic*) if every vertex lies on a cycle of every length (every length) from m to $|V(G)|$. Jwo *et al.* showed that an S_n contains a cycle of every even length from 6 to $n!$ [10]. Since an S_n is vertex symmetric [1], it is 6-weak-vertex-bipancyclic. This work further investigates the m-weak-vertex-pancyclicity of $S_{n,k}$ graphs.

The rest of this paper is organized as follows. Section 2 formally defines the $S_{n,k}$ in terms of graph (or interconnection networks). Section 3 demonstrates how to embedded cycles in an $S_{n,k}$ graph is proved. Conclusions are finally drawn in Section 4.

2 Background and Notations

This section formally presents the structure of the (n, k)-star graph and studies some basic properties of it.

For simplicity, let $\langle n\rangle = \{1, 2, \ldots, n\}$, $\langle k\rangle = \{1, 2, \ldots, k\}$, and two positive integers n and k satisfy $1 \le k \le n–1$. An $S_{n,k}$ is specified by two integers n and k, where $1 \le k \le n–1$. The vertex set of it is denoted by $\{p_1p_2\ldots p_k \mid p_i \in \langle n\rangle$ and $p_i \ne p_j$ for $i \ne j\}$. The adjacency is defined as follows: $p_1p_2\ldots p_i\ldots p_k$ is adjacent to (1) $p_ip_2\ldots p_1\ldots p_k$ through an edge of dimension i, where $2 \le i \le k$, and (2) $xp_2\ldots p_k$ through dimension 1, where $x \in \langle n\rangle - \{p_q \mid 1 \le q \le k\}$.

The edges of type (1) are referred to as i-edges and the two endvertices are i-neighbors to each other. The edges of type (2) are referred to as 1-edges, the two endvertices are 1-neighbors to each other [11]. The structure of an $S_{4,2}$ is shown in Fig. 1.

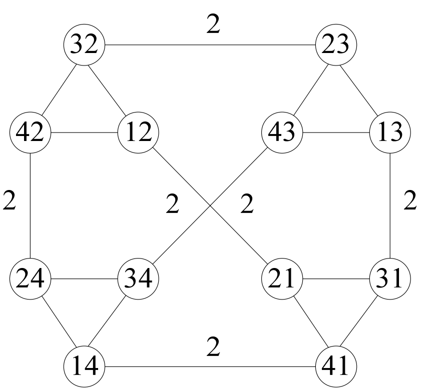

Fig. 1. The structure of an $S_{4,2}$.

Let $S_{n−1,k−1}(\omega)$ denote a subgraph of an

$S_{n,k}$, induced by all the vertices with the same last symbol ω, for some $1 \le \omega \le n$. By the structure of (n, k)-star graph, 1-edges and i-edges still remain what they are in $S_{n-j,k-j}$ for $1 \le j \le k-1$ and $1 \le i \le k-j$.

An $S_{n,k}$ can be formed by interconnecting n $S_{n-1,k-1}$'s. Fig. 1 shows that $S_{4,2}$ can be viewed as an interconnection of $S_{3,1}(\omega)$'s, $1 \le \omega \le 4$, through 2-edges. That is, an $S_{n,k}$ can be decomposed into $S_{n-1,k-1}$'s along any dimension i, and it can also be decomposed into n vertex-disjoint $S_{n-1,k-1}$'s in $k-1$ different ways by fixing the symbol in any position i, $2 \le i \le k$ [4], [5]. This decomposition can be recursively carried out on each $S_{n-1,k-1}$ to obtain smaller subgraphs.

A graph is said to be *vertex* (*edge*) *symmetric* if for every pair of vertices (edges), a and b, there exists an automorphism of the graph that maps a into b. The $S_{n,k}$ is undirected and vertex symmetric [4], but two different types of edges (1-edge and i-edge for $2 \le i \le k$) in the $S_{n,k}$ prevent it from being edge symmetric. As shown in Fig. 1, each 2-edge belongs to a cycle of length at least 6, but each 1-edge may be within a cycle of length 3. However, there is still conditional edge-symmetric which was proved as follows.

Lemma 1 [5]. In an $S_{n,k}$, every 1-edge is edge-symmetric with any other 1-edge.

Lemma 2 [5]. In an $S_{n,k}$, every i-edge is edge-symmetric with any other i-edge, $2 \le i \le k$.

According to Lemmas 1 and 2, an $S_{n,k}$ is j-edge-symmetric for $1 \le j \le k$. Some interesting topological properties of an $S_{n,k}$ used in Section 3 are stated as follows.

Lemma 3 [12]. In an $S_{n,k}$, a cycle has a length at least 6 if it contains one i-edge, $2 \le i \le k$.

Lemma 4 [11]. The $S_{n,1}$ is isomorphic to the K_n, which is a complete graph.

Lemma 5 [11]. The $S_{n,n-1}$ is isomorphic to the S_n.

Lemma 6 [4]. There are $(n-2)!/(n-k)!$ k-edges between any two subgraphs $S_{n-1,k-1}(\alpha)$ and $S_{n-1,k-1}(\beta)$ in an $S_{n,k}$; each of these vertices in an $S_{n-1,k-1}(\alpha)$ is connected to exactly one vertex in $S_{n-1,k-1}(\beta)$, where $1 \le \alpha, \beta \le n$ and $\alpha \ne \beta$.

3 Cycle Embedding

Lemma 4 indicates that $S_{n,1}$ is vertex-pancyclic for trivial. Chiang already proved that S_n, which is isomorphic to $S_{n,n-1}$ (Lemma 5), contains a cycle of every even length from 6 to $n!$ [10]. Since S_n is vertex symmetric, $S_{n,n-1}$ is 6-vertex-bipancyclic. In the following, the vertex-pancyclicity of $S_{n,2}$ is first shown, and then the vertex-pancyclicity of $S_{n,k}$ is proved by induction on k for $3 \le k \le n-4$. Finally, the two special cases of $S_{n,n-2}$ and $S_{n,n-3}$ is also discussed.

In the remainder of this section, let $p \rightarrow_{(j)} q$ denote that vertices p and q is connected by a j-edge. Moreover, $p \Rightarrow q$ is defined as a spanning path in an $S_{n-1,k-1}$ from p to q. Since an $S_{3,2}$ is isomorphic to S_3, which is 6-weak-vertex-bipancyclic, only the pancyclicity or weak-pancyclicity of each $S_{n,2}$ for $n \ge 4$ is discussed in Lemmas 7 and 8.

Lemma 7. An $S_{n,2}$ is vertex-pancyclic for $n \ge 6$. Moreover, each cycle of an available length in an $S_{n,2}$ can contain a desired 1-edge.

Proof. $S_{n,2}$ is composed of n $S_{n-1,1}(\omega)$'s, where $1 \le \omega \le n$. Lemma 4 reveals that an $S_{n-1,1}(\omega)$ is isomorphic to a K_{n-1}, which has cycles of length ranged from 3 to $n-1$. Next, a cycle of length ranged from n to $n(n-1)$ contained in at least three $S_{n-1,1}(\omega)$'s is build step by step as follows.

1. Adopt 1-edge and 2-edge to connect the first three $S_{n-1,1}(\omega)$'s. Let $p = p_1p_2$ be the source vertex in the $S_{n,2}$ and $x_t \in \langle n \rangle - \{p_q \mid 1 \le q \le 2\}$, where $3 \le t \le n$. According to Lemma 3, a cycle in an $S_{n,2}$

containing a 2-edge has length at least 6. Lemma 6 indicates that there exists one 2-edge between any two $S_{n-1,1}(\omega)$'s. A cycle of length 6 can be built as $p \rightarrow_{(2)} p_2p_1 \rightarrow_{(1)} x_3p_1 \rightarrow_{(2)} p_1x_3 \rightarrow_{(1)} p_2x_3 \rightarrow_{(2)} x_3p_2 \rightarrow_{(1)} p$.

2. The constructed cycle visits $S_{n-1,1}(p_2)$, $S_{n-1,1}(p_1)$, and $S_{n-1,1}(x_3)$. Because a complete graph (such as an $S_{n-1,1}$) is vertex-pancyclic, a cycle of any length containing any edge in it could be constructed easily. Adopting the 1-edge (p_2p_1, x_3p_1) in the $S_{n-1,1}(p_1)$ to build a cycle of length $n-4$ in it; a cycle of length n can be constructed after removing edge (p_2p_1, x_3p_1) as shown in Fig. 2. Notably, in the remainder of this paper, each of Figs. 2–8 explicitly indicates the number of vertices in each subpath of a cycle.

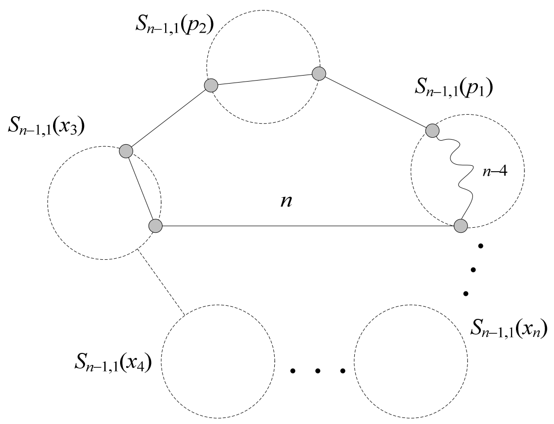

Fig. 2. A cycle of length n in an $S_{n,2}$.

3. By adding vertices in these three $S_{n-1,1}$'s, which are excluded by the cycle constructed in Step 2, into the cycle one-by-one, a cycle of length ranged from $n+1$ to $3n-3$ can be established. Fig. 3 illustrates the cycle of length $3n-3$.

4. To add the $S_{n-1,1}(x_4)$, for constructing a cycle of length $3n-2$, remove the path in $S_{n-1,1}(p_2)$ and $S_{n-1,1}(x_3)$, a new cycle of length $n+5$ containing four subgraphs is form as $p \rightarrow_{(2)} p_2p_1 \Rightarrow_{(1)} x_3p_1 \rightarrow_{(2)} p_1x_3 \rightarrow_{(1)} x_4x_3 \rightarrow_{(2)} x_3x_4 \rightarrow_{(1)} p_2x_4 \rightarrow_{(2)} x_4p_2 \rightarrow_{(1)} p$.

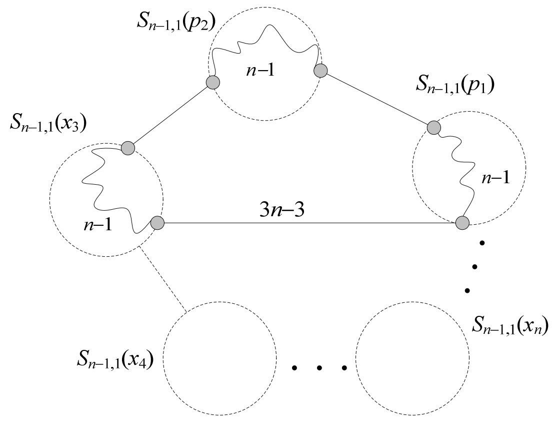

Fig. 3. A cycle of length $3n-3$ in an $S_{n,2}$.

5. Build cycles of length $n-1$ and $n-2$ with the edge (p_1x_3, x_4x_3) in $S_{n-1,1}(x_3)$ and the edge (x_3x_4, p_2x_4) in $S_{n-1,1}(x_4)$, a cycle of length $3n-2$ can be obtained after eliminating the edges (p_1x_3, x_4x_3) and (x_3x_4, p_2x_4) as shown in Fig. 4. Referring to Steps 3, a cycle of length ranged from $3n-1$ to $4n-4$ can be established.

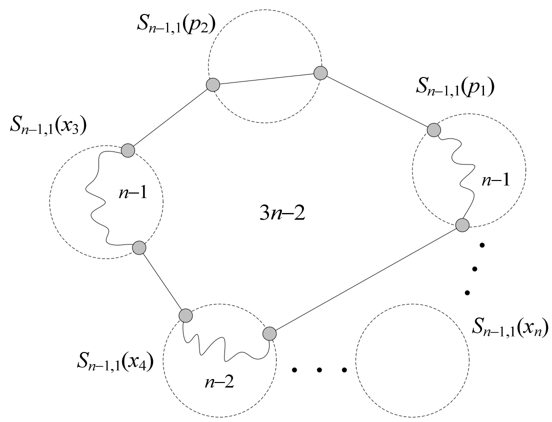

Fig. 4. A cycle of length $3n-2$ in an $S_{n,2}$.

6. Refer to Steps 3 to 5 to build a cycle of length ranged from $4n-3$ to $n(n-1)$. The longest cycle is constructed as $p \rightarrow_{(2)}$

$p_2p_1 \Rightarrow x_3p_1 \rightarrow_{(2)} p_1x_3 \Rightarrow x_4x_3 \rightarrow_{(2)} x_3x_4 \Rightarrow \ldots \rightarrow_{(2)} x_{n-1}x_n \Rightarrow p_2x_n \rightarrow_{(2)} p$.

Obviously, all constructed cycles contain 1-edges. Therefore, according to Lemma 1, each constructed cycle in an $S_{n,2}$ can contain a desired 1-edge. □

Lemma 8. $S_{4,2}$ and $S_{5,2}$ are both 6-weak-vertex-pancyclic.

Proof. The proof is similar to that of Lemma 7. An $S_{4,2}$ ($S_{5,2}$) is composed of 4 $S_{3,1}$'s (5 $S_{4,1}$'s) and has a cycle of length l for $l = 3$ ($l = 3, 4$). Moreover, Lin and Duh showed that the length of a cycle containing an i-edge is at least 6 [12]. Therefore, the $S_{4,2}$ ($S_{5,2}$) has no cycles of length 4 (4 and 5). Let $p = p_1p_2$ be the source vertex in the $S_{4,2}$ ($S_{5,2}$) and $x_t \in \langle n\rangle - \{p_q \mid 1 \le q \le 2\}$, where $3 \le t \le 4$ ($3 \le t \le 5$). Three basic cycles of length 6, 8 and 10 can be built in $S_{4,2}$ and $S_{5,2}$ as follows.

1. Cycle 1: $p \rightarrow_{(2)} p_2p_1 \rightarrow_{(1)} x_3p_1 \rightarrow_{(2)} p_1x_3 \rightarrow_{(1)} p_2x_3 \rightarrow_{(2)} x_3p_2 \rightarrow_{(1)} p$.
2. Cycle 2: $p \rightarrow_{(2)} p_2p_1 \rightarrow_{(1)} x_3p_1 \rightarrow_{(2)} p_1x_3 \rightarrow_{(1)} x_4x_3 \rightarrow_{(2)} x_3x_4 \rightarrow_{(1)} p_2x_4 \rightarrow_{(2)} x_4p_2 \rightarrow_{(1)} p$.
3. Cycle 3: $p \rightarrow_{(2)} p_2p_1 \rightarrow_{(1)} x_3p_1 \rightarrow_{(2)} p_1x_3 \rightarrow_{(1)} x_4x_3 \rightarrow_{(2)} x_3x_4 \rightarrow_{(1)} x_5x_4 \rightarrow_{(2)} x_4x_5 \rightarrow_{(1)} p_2x_5 \rightarrow_{(2)} x_5p_2 \rightarrow_{(1)} p$.

First, expand Cycle 1 by adding vertices one-by-one in the $S_{4,2}$ ($S_{5,2}$) to construct a cycle of length ranged from 7 to 9 (12). Second, expand Cycle 2 in the $S_{4,2}$ ($S_{5,2}$) to construct a cycle of length ranged form 9 to $|V(S_{4,2})|$=12 (16). Finally, expand Cycle 3 by including the $S_{4,1}(x_5)$ of the $S_{5,2}$ to built a cycle of length l for $11 \le l \le |V(S_{5,2})| = 20$. □

Restated, an $S_{3,2}$ is a S_3. According to Lemma 5 and [10], an $S_{3,2}$ is 6-weak-vertex-bipancyclic. Lemma 8 indicates that an $S_{4,2}$ ($S_{5,2}$) is 6-weak-vertex-pancyclic. An $S_{4,2}$ ($S_{5,2}$) is the induction base for proving the 6-weak-vertex-pancyclicity of an $S_{n,n-2}$ ($S_{n,n-3}$) for $n \ge 5$ ($n \ge 6$). The 6-weak-vertex-pancyclicity of an $S_{n,n-2}$ ($S_{n,n-3}$) is demonstrated in Lemma 11. Hsu *et al.* showed that an $S_{n,k}$ has a spanning cycle of length $|V(S_{n,k})|$ [8]. Moreover, an $S_{n,k}$ is composed of $n!/(n-k+1)!$ $S_{n-k+1,1}$'s (or K_{n-k+1}). Therefore, by Lemmas 1 and 6, any 1-edge can be contained in a spanning cycle of an $S_{n,k}$.

Lemma 9. Every spanning cycle of an $S_{n,k}$, $n > k \ge 2$, including a given 1-edge contains another 1-edge which is endvertex-disjoint to the given one.

Proof. Since $n > k \ge 2$, there exist n (≥ 3) $S_{n-1,k-1}(\omega)$'s in the $S_{n,k}$, where $1 \le \omega \le n$. Thus, at least $n-1$ (≥ 2) 1-edges are endvertex-disjoint to the given 1-edge. □

According to Lemma 9, two disjoint-paths containing all vertices of an $S_{n-1,k-1}(\omega)$ can be built for $n > k \ge 3$. This property is used to constructed a cycle of length at least $(n-k+2)|V(S_{n-1,k-1})|+1$ in an $S_{n,k}$ as described in Step 6 of the proof of Lemma 10.

Since an $S_{5,1}$ is a K_5, only the vertex-pancyclicity of an $S_{n,k}$ for $3 \le k \le n-4$ and $n \ge 6$ is presented in the following.

Lemma 10. An $S_{n,k}$ is vertex-pancyclic for $3 \le k \le n-4$ and $n \ge 6$. Moreover, each cycle of an available length in an $S_{n,k}$ can contain a desired 1-edge.

Proof. Recall that an $S_{n,1}$ is an K_n and vertex-pancyclic. According to Lemma 7, $S_{\delta,2}$ is vertex-pancyclic and each constructed cycle can contain a desired 1-edge in it for $6 \le \delta \le n$. This lemma is proved by induction on n and k, and adopts an $S_{\delta,2}$ as the basis, where $6 \le \delta \le n$. Let $p = p_1p_2\ldots p_k$, $x_t \in \langle n\rangle - \{p_q \mid 1 \le q \le k\}$ and $N = |V(S_{n-1,k-1})|$, where $k+1 \le t \le n$, $l = (n-k+2)N$ ($l = (n-k+1)N$) for

k is even (odd), and $\eta = n$ ($\eta = n-1$) for k is even (odd). Assume that an $S_{n-1,k-1}$ is vertex-pancyclic and each constructed cycle can contain a desired 1-edge in the $S_{n-1,k-1}$. Hence, constructing a cycle of length smaller than or equal to N in an $S_{n,k}$ is trivial. Only the cycle of length ranged from N+1 to nN should be discussed as follows. Notably, the constructed cycle must contain at least three $S_{n-1,k-1}$'s in the $S_{n,k}$.

1. According to Lemma 3, a cycle in an $S_{n,k}$ containing a k-edge has length at least 6. Lemma 6 indicates that there exist $(n-2)!/(n-k)!$ k-edges between any two $S_{n-1,k-1}(\omega)$'s, where $1 \le \omega \le n$. A cycle of length 6 containing 2 vertices in each of the included $S_{n-1,k-1}(\omega)$'s is built with 1-edges and k-edges as

$$
\begin{aligned}
p &\rightarrow_{(k)} p_k p_2 p_3 \ldots p_{k-1} p_1 \\
&\rightarrow_{(1)} x_{k+1} p_2 p_3 \ldots p_{k-1} p_1 \\
&\rightarrow_{(k)} p_1 p_2 \ldots p_{k-1} x_{k+1} \\
&\rightarrow_{(1)} p_k p_2 p_3 \ldots p_{k-1} x_{k+1} \\
&\rightarrow_{(k)} x_{k+1} p_2 p_3 \ldots p_k \\
&\rightarrow_{(1)} p.
\end{aligned}
$$

2. Lemma 1 indicates that an $S_{n-1,k-1}(\omega)$ is 1-edge-symmetric for $1 \le \omega \le n$. Therefore, a cycle of length at most N can be constructed in the $S_{n-1,k-1}(\omega)$ and the cycle can contains any 1-edge. Build a cycle of length N–3 in the $S_{n-1,k-1}(p_1)$ including the specified 1-edge $(p_k p_2 p_3 \ldots p_{k-1} p_1,\ x_{k+1} p_2 p_3 \ldots p_{k-1} p_1)$. After removing the specified 1-edge, a cycle of length N+1, can be constructed by expanding the cycle constructed in Step 1.

3. The cycle constructed in Step 2 can be increased by including vertices in $S_{n-1,k-1}(p_1)$, $S_{n-1,k-1}(x_{k+1})$, and $S_{n-1,k-1}(p_k)$, which are excluded in Step 2. Finally, a cycle of length ranged from N+2 to 3N can be obtained.

4. To add the $S_{n-1,k-1}(x_{k+2})$ for constructing a cycle of length 3N+1, remove the path in $S_{n-1,k-1}(p_k)$ and $S_{n-1,k-1}(x_{k+1})$. A cycle of length N+6 is first formed as

$$
\begin{aligned}
p &\rightarrow_{(k)} p_k p_2 p_3 \ldots p_{k-1} p_1 \\
&\Rightarrow x_{k+1} p_2 p_3 \ldots p_{k-1} p_1 \\
&\rightarrow_{(k)} p_1 p_2 \ldots p_{k-1} x_{k+1} \\
&\rightarrow_{(1)} x_{k+2} p_2 p_3 \ldots p_{k-1} x_{k+1} \\
&\rightarrow_{(k)} x_{k+1} p_2 p_3 \ldots p_{k-1} x_{k+2} \\
&\rightarrow_{(1)} p_k p_2 p_3 \ldots p_{k-1} x_{k+2} \\
&\rightarrow_{(k)} x_{k+2} p_2 p_3 \ldots p_k \\
&\rightarrow_{(1)} p.
\end{aligned}
$$

Second, build two cycles of lengths N and N–1 in $S_{n-1,k-1}(p_k)$ and $S_{n-1,k-1}(x_{k+1})$ containing the 1-edges $(x_{k+2} p_2 p_3 \ldots p_k,\ p)$ and $(p_1 p_2 \ldots p_{k-1} x_{k+1},\ x_{k+2} p_2 p_3 \ldots p_{k-1} x_{k+1})$, respectively. A cycle of length 3N+1 can be obtained after eliminating the two specified 1-edges.

5. Referring to Steps 3 and 4, a cycle of length ranged from 3N+2 to l can be similarly constructed. Moreover, the constructed cycle only use 1-edges and k-edges to connect $S_{n-1,k-1}(p_1)$, $S_{n-1,k-1}(p_k)$, and $S_{n-1,k-1}(x_t)$ for $k+1 \le t \le \eta$. Notably, the $S_{n-1,k-1}(x_n)$ is not joined in this step when k is odd. The cycle of length l is constructed as follows and shown in Fig. 5.

$$
\begin{aligned}
p &\rightarrow_{(k)} p_k p_2 p_3 \ldots p_{k-1} p_1 \\
&\Rightarrow x_{k+1} p_2 p_3 \ldots p_{k-1} p_1 \\
&\rightarrow_{(k)} p_1 p_2 \ldots p_{k-1} x_{k+1}
\end{aligned}
$$

$\Rightarrow x_{k+2}p_2p_3\ldots p_{k-1}x_{k+1}$
$\rightarrow_{(k)} x_{k+1}p_2p_3\ldots p_{k-1}x_{k+2}$
$\Rightarrow\ldots$
$\Rightarrow x_{k+s}p_2p_3\ldots p_{k-1}x_{k+s-1}$
$\rightarrow_{(k)} x_{k+s-1}p_2p_3\ldots p_{k-1}x_{k+s}$
$\Rightarrow\ldots$
$\rightarrow_{(k)} x_{\eta-1}p_2p_3\ldots p_{k-1}x_{\eta}$
$\Rightarrow p_kp_2p_3\ldots p_{k-1}x_{\eta}$
$\rightarrow_{(k)} x_{\eta}p_2p_3\ldots p_k$
$\Rightarrow p$, where $1 \le s \le \eta-k-1$.

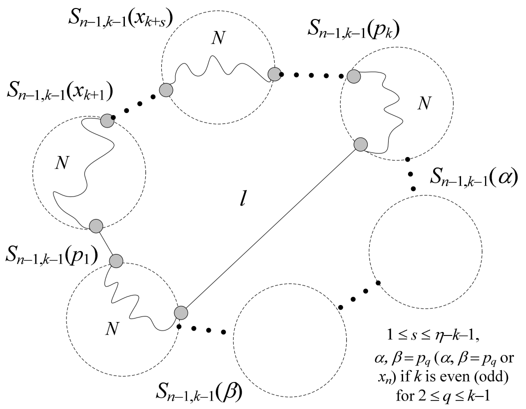

Fig. 5. A cycle of length l.

Let $R = \{\gamma \mid \gamma = p_q$ for $2 \le q \le k-1\}$ ($R = \{\gamma \mid \gamma = p_q$ or x_n for $2 \le q \le k-1\}$) if k is even (odd). Significantly, every $S_{n-1,k-1}(\gamma)$ does not included in the constructed cycle.

6. Rebuild a path containing $N-3$ vertices from $p_kp_2p_3\ldots p_{k-1}p_1$ to $x_{k+1}p_2p_3\ldots p_{k-1}p_1$ in $S_{n-1,k-1}(p_1)$. According to Lemma 9, there exists a 1-edge (u, v) other than the 1-edge $(x_{\eta}p_2p_3\ldots p_k, p)$ in the current cycle and the $S_{n-1,k-1}(p_k)$ such that vertex u (v) has a k-neighbor in the $S_{n-1,k-1}(\alpha)$ ($S_{n-1,k-1}(\beta)$), where $\alpha, \beta \in R$ and $\alpha \ne \beta$. Remove α and β from the set R. A cycle of length $l+1$ can be constructed by removing the 1-edge (u, v) and including two vertices in each of $S_{n-1,k-1}(\alpha)$ and $S_{n-1,k-1}(\beta)$ as shown in Fig. 6.

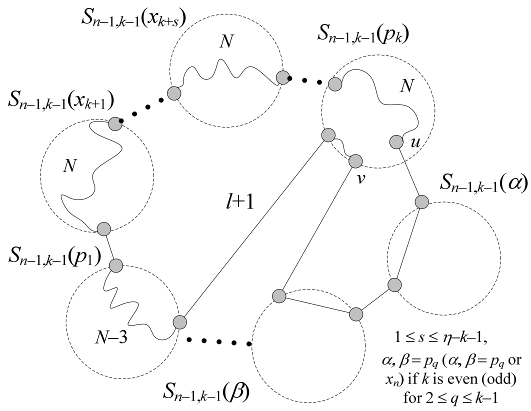

Fig. 6. A cycle of length $l+1$.

7. By adding vertices one-by-one in $S_{n-1,k-1}(p_1)$, $S_{n-1,k-1}(\alpha)$, and $S_{n-1,k-1}(\beta)$, which are excluded by the cycle constructed in Step 6, a cycle of length ranged from $l+2$ to $l+2N$ can be obtained.

8. Referring to Steps 6 and 7, a cycle of length ranged from $l+2N+1$ to nN can be similarly built by removing 2 elements from the set R every time until it is empty.

Significantly, all constructed cycles contain 1-edges. Therefore, according to Lemma 1, each constructed cycle in an $S_{n,k}$ can contain a desired 1-edge. □

Lemma 11. An $S_{n,n-2}$ ($S_{n,n-3}$) is 6-weak-vertex-pancyclic for $n \ge 5$ ($n \ge 6$). Moreover, each cycle of an available length in an $S_{n,n-2}$ ($S_{n,n-3}$) can contain a desired 1-edge.

Proof. This proof is similar to that of Lemma 10. Because an $S_{n,n-2}$ ($S_{n,n-3}$) has no cycles of length 4 (4 and 5), some construction should be modified as follows. Let $k = n-2$ ($k = n-3$) for the $S_{n,n-2}$ ($S_{n,n-3}$) and N represents the number of vertices in an $S_{n-1,k-1}$.

1. Lemma 8 indicates that an $S_{n-1,k-1}$ only

contains every cycle of length l for $6 \le l \le N$. In other words, a cycle of length ranged from 6 to N can be constructed and it can contain a desired 1-edge.

2. Referring to Steps 1 to 3 of Lemma 10, a cycle of length ranged from N+1 to N+4 can be built similarly. Notably, each vertex in the constructed cycle is contained in $S_{n-1,k-1}(p_1)$, $S_{n-1,k-1}(p_k)$, or $S_{n-1,k-1}(x_{k+1})$.

3. First, rebuild a path of length N–3 in the $S_{n-1,k-1}(p_1)$, and construct a cycle of length 6 in the $S_{n-1,k-1}(p_k)$. Notably, an $S_{4,2}$ (or an $S_{5,2}$) is a subgraph of an $S_{n-1,k-1}(\omega)$, where $1 \le \omega \le n$, and it contains a cycle of length 6 by Lemma 8. Second, After removing the specified 1-edge in the $S_{n-1,k-1}(p_k)$, a cycle of length N+5 can be built as shown in Fig. 7. Referring to Step 3 of the proof of Lemma 10, a cycle of length ranged from N+6 to 3N can be constructed.

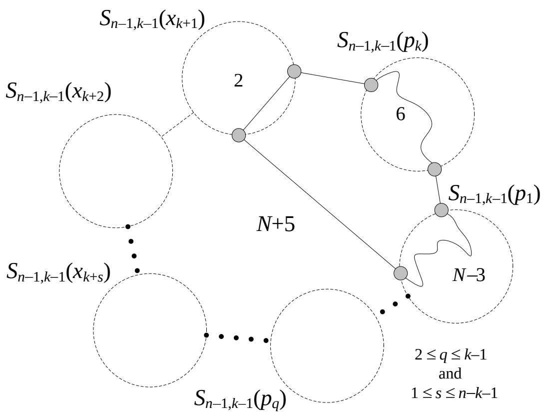

Fig. 7. A cycle of length N+5.

4. According to Step 4 of the proof of Lemma 10, a cycle of length 3N+1 can be similarly established as shown in Fig. 8. Significantly, every time expanding the cycle by including a subgraph, a path of length 6 is constructed in the just included subgraph and a path of length N–5 is rebuilt in one of the other subgraphs containing some vertices of the cycle.

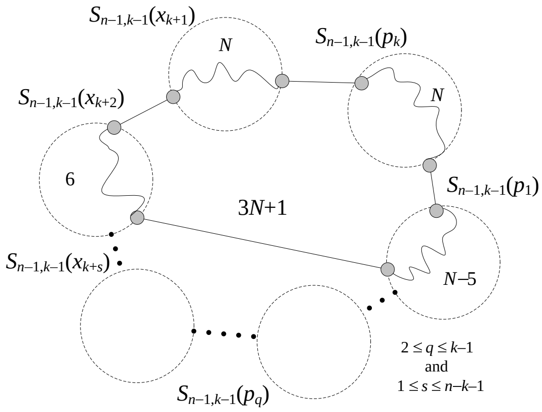

Fig. 8. A cycle of length 3N+1.

5. Referring to Step 3 of the proof of Lemma 10, a cycle of length ranged from 3N+2 to 4N can be similarly established by including vertices.

6. Referring to Steps 5 to 8 of the proof of Lemma 10 but the paths in just included two subgraphs contain 6 and N–5 vertices, respectively, a cycle of length ranged from 4N+1 to nN can be similarly built.

Restated, all constructed cycles contain 1-edges. Therefore, according to Lemma 1, each constructed cycle in an $S_{n,n-2}$ ($S_{n,n-3}$) can contain a desired 1-edge. □

Theorem 12. An $S_{n,k}$ is

$$\begin{cases} \text{vertex-pancyclic} & \text{if } 1 \le k \le n-4 \\ \text{6-weak-vertex-pancyclic} & \text{if } n-3 \le k \le n-2 \\ \text{6-weak-vertex-bipancyclic} & \text{if } k = n-1. \end{cases}$$

Moreover, each cycle of an available length in an $S_{n,k}$ can contain a desired 1-edge.

4 Conclusion

Although an S_n in general is not

vertex-pancyclic, the weak-pancyclicity of an S_n is revealed in this work. Chang and Kim already showed that an S_n (or $S_{n,n-1}$) is 6-weak-vertex-bipancyclic. Trivially, a K_n is vertex-pancyclic. This work shows that an $S_{n,k}$ is vertex-pancyclic if $2 \le k \le n-4$ and $n \ge 6$, and an $S_{n,k}$ is 6-weak-vertex-pancyclic if $n-3 \le k \le n-2$. Thus, an $S_{n,k}$ is vertex-pancyclic for $1 \le k \le n-4$ and $n \ge 6$, 6-weak-vertex-pancyclic for $n-3 \le k \le n-2$, or 6-weak-vertex-bipancyclic if $k = n-1$. Significantly, each constructed cycle of an available length in an $S_{n,k}$ can contain a desired 1-edge.

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