On the Super Spanning Properties of Binary Wrapped Butterfly Graphs

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Abstract

A k-container $C_k(u,v)$ of a graph G is a set of k internally vertex-disjoint paths joining vertices u and v. It becomes a k^*-container if every vertex of G is passed by a certain path of $C_k(u,v)$. A graph G is said to be k^*-connected if there exists a k^*-container between any two vertices of G. A graph G with connectivity κ is super spanning connected if it is i^*-connected for every $1 \leq i \leq \kappa$. A bipartite graph G is k^*-laceable if there exists a k^*-container between any two vertices u and v from different partite sets of G. A bipartite graph G with connectivity κ is super spanning laceable if it is i^*-laceable for all $1 \leq i \leq \kappa$. In this paper, we show the n-dimensional binary wrapped butterfly graph is super spanning connected (resp. super spanning laceable) if n is odd (resp. even).

Keywords: Container; Butterfly graph; Hamiltonian connected; Hamiltonian laceable; Super spanning connected; Super spanning laceable

1 Introduction

A multiprocessor/communication *interconnection network* is usually represented by a graph, in which the vertices correspond to processors and the edges correspond to connections or communication links. Hence, we use the terms, graphs and networks, interchangeably. Among various kinds of network topologies, the wrapped *butterfly network* is very suitable for VLSI implementation and parallel computing and thus its topological properties have been widely discussed in [1, 4–7, 11, 12, 14]. For example, embedding of *rings*, *linear arrays*, and *binary trees* into a butterfly network was addressed in literature [5, 6, 12, 14].

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Throughout this paper, we concentrate on loopless undirected graphs. For the graph definitions and notations we follow the ones defined in [2]. A graph G consists of a finite nonempty set $V(G)$ and a subset $E(G)$ of $\{(u,v) \mid (u,v)$ is an unordered pair of $V(G)\}$. The set $V(G)$ is called the *vertex set* of G and $E(G)$ is called the *edge set*. Two vertices u and v of G are adjacent if $(u,v) \in E(G)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a nonempty subset of $V(G)$. The subgraph *induced* by S is the subgraph of G with its vertex set S and with its edge set which consists of those edges joining any two vertices in S. We use $G - S$ to denote the subgraph of G induced by $V(G) - S$. Analogously, the subgraph generated by a nonempty subset $F \subseteq E(G)$ is the subgraph of G with its edge set F and with its vertex set consisting of those vertices of G incident with at least one edge of F. We use $G - F$ to denote the subgraph of G with vertex set $V(G)$ and edge set $E(G) - F$.

A *path* P of length k joining vertex x to vertex y in a graph G is a sequence of distinct vertices $\langle v_1, v_2, \ldots, v_{k+1}\rangle$ such that $x = v_1$, $y = v_{k+1}$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \leq i \leq k$. For convenience, we write P as $\langle v_1, \ldots, v_i, Q, v_j, \ldots, v_{k+1}\rangle$ where $Q = \langle v_i, v_{i+1}, \ldots, v_j\rangle$. Note that we allow Q to be a path of length zero. For $i \geq 1$, the i-th vertex of P is denoted by $P(i)$; i.e., $P(i) = v_i$. Moreover, we use P^{-1} to denote the path $\langle v_{k+1}, v_k, \ldots, v_1\rangle$. Let $V(P) = \{v_1, v_2, \ldots, v_{k+1}\}$ and $I(P) = V(P) - \{v_1, v_{k+1}\}$. A set of k paths $P_1, \ldots, P_k$ are *internally vertex-disjoint* (or *disjoint* for short) if $I(P_i) \cap I(P_j) = \phi$ for any $i \neq j$. A *cycle* is a path with at least three vertices such that the last vertex is adjacent to the first one. For clarity, a cycle of length k is represented by $\langle v_1, v_2, \ldots, v_k, v_1\rangle$. A path of a graph G is a *hamiltonian path* if it spans G. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two distinct vertices of G. Similarly, a *hamiltonian cycle* of a graph G is a cycle that traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle.

The *degree* of a vertex u in G is the number of edges incident to u. A graph G is *k-regular* if all its vertices have the same degree k. The *connectivity* of a graph G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. A *k-container* $C_k(u,v)$ of G between vertices u and v is a set of k internally vertex-disjoint paths joining u and v. Suppose that $\kappa(G) = k$. It follows from Menger's theorem [9] that there exists a k-container of G between any two distinct vertices. In this paper, a k-container $C_k(u,v)$ is a *k^*-container* if it contains all vertices of G. A graph G is *k^*-connected* if there exists a k^*-container between any two distinct vertices. In particular, G is 1^*-connected if and only if it is hamiltonian-connected. Moreover, G is 2^*-connected if it is hamiltonian. The *spanning connectivity* of a graph G, $\kappa^*(G)$, is defined as the largest integer m such that G is i^*-connected for all $1 \le i \le m$. A graph G is *super spanning connected* if $\kappa^*(G) = \kappa(G)$. Recently, a number of networks had been shown to be super spanning connected [8, 13].

A graph G is *bipartite* if its vertex set can be partitioned into two disjoint subsets V_0 and V_1 such that every edge joins a vertex of V_0 and a vertex of V_1. A bipartite graph is *k^*-laceable* if there exists a k^*-container between any two vertices from different partite sets. Note that a 1^*-laceable graph is also known as a *hamiltonian-laceable* graph [10] and a bipartite graph is 2^*-laceable if and only if it is hamiltonian. Similarly, the *spanning laceability* of a hamiltonian laceable graph G, $\kappa^*_L(G)$, is the largest integer m such that G is i^*-laceable for every $1 \le i \le m$. A bipartite graph G is *super spanning laceable* if $\kappa^*_L(G) = \kappa(G)$. Likewise, a number of networks had been shown to be super spanning laceable [3, 8].

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ denote the set of integers modulo n. The n-dimensional binary *wrapped butterfly network* (or *butterfly network* for short) $BF(n)$ is a graph with $\mathbb{Z}_n \times \mathbb{Z}_2^n$ as vertex set. Each vertex is labeled by a two-tuple $\langle \ell, a_0 \ldots a_{n-1}\rangle$ with a level $\ell \in \mathbb{Z}_n$ and an n-digit binary string $a_0 \ldots a_{n-1} \in \mathbb{Z}_2^n$. A level-$\ell$ vertex $\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1}\rangle$ is adjacent to two vertices, $\langle (\ell+1)_{\bmod n}, a_0 \ldots a_\ell \ldots a_{n-1}\rangle$ and $\langle (\ell-1)_{\bmod n}, a_0 \ldots a_{\ell-1} \ldots a_{n-1}\rangle$, by *straight edges*, and is adjacent to another two vertices, $\langle (\ell+1)_{\bmod n}, a_0 \ldots a_{\ell-1}\overline{a}_\ell a_{\ell+1} \ldots a_{n-1}\rangle$ and $\langle (\ell-1)_{\bmod n}, a_0 \ldots a_{\ell-2}\overline{a}_{\ell-1} a_\ell \ldots a_{n-1}\rangle$, by *cross edges*. More formally, the edges of $BF(n)$ can be defined in terms of four generators g, g^{-1}, f, and f^{-1} as follows:

$$g(\langle \ell, a_0 \ldots a_{n-1}\rangle) = \langle (\ell+1)_{\bmod n}, a_0 \ldots a_{n-1}\rangle,$$

$$g^{-1}(\langle \ell, a_0 \ldots a_{n-1}\rangle) = \langle (\ell-1)_{\bmod n}, a_0 \ldots a_{n-1}\rangle,$$

$$f(\langle \ell, a_0 \ldots a_\ell \ldots a_{n-1}\rangle) = \langle (\ell+1)_{\bmod n}, a_0 \ldots \overline{a}_\ell \ldots a_{n-1}\rangle,$$

and

$$\begin{aligned} &f^{-1}(\langle \ell, a_0 \ldots a_{\ell-1} \ldots a_{n-1}\rangle \\ &= \langle (\ell-1)_{\bmod n}, a_0 \ldots \overline{a}_{\ell-1} \ldots a_{n-1}\rangle \end{aligned}$$

where $\overline{a}_\ell \equiv a_\ell + 1 \pmod 2$. Throughout this paper, a level-ℓ edge of $BF(n)$ is an edge that joins a level-ℓ vertex and a level-$(\ell+1)_{\bmod n}$ vertex. To avoid the degenerate case, we assume $n \ge 3$ throughout this paper. So $BF(n)$ is 4-regular. Moreover, $BF(n)$ is bipartite if and only if n is even. Figure 1(a) depicts the structure of $BF(3)$ and Figure 1(b) is the isomorphic structure of $BF(3)$ with level-0 replication for easy visualization.

According to [14], $BF(n)$ is 1^*-connected and 2^*-connected (resp. 1^*-laceable and 2^*-laceable) if n is odd (resp. even). In this paper, we show that $BF(n)$ is 3^*-connected and 4^*-connected (resp. 3^*-laceable and 4^*-laceable) if n is odd (resp. even). The rest of the paper is organized as follows. Section 2 introduces the nearly recursive construction of $BF(n)$, which was proposed by Wong [14]. Section 3 provides the useful lemmas to prove the main results. Since the proof of the main theorem is rather long, it is broken into several lemmas in Section 4 and Section 5. For the sake of clarity, the detailed proofs of several lemmas are described in Appendix. Finally, the future work is discussed in Section 6.

2 Nearly recursive construction of $BF(n)$

Let $n \ge 3$. For any $\ell \in \mathbb{Z}_n$ and $i \in \mathbb{Z}_2$, we use $BF^i_\ell(n)$ to denote the subgraph of $BF(n)$ induced by $\{\langle h, a_0 \ldots a_{n-1}\rangle \mid h \in \mathbb{Z}_n, a_\ell = i\}$. Obviously, $BF^i_{\ell_1}(n)$ is isomorphic to $BF^j_{\ell_2}(n)$ for any $i, j \in \mathbb{Z}_2$ and $\ell_1, \ell_2 \in \mathbb{Z}_n$. Moreover, $\{BF^i_\ell(n) \mid i \in \mathbb{Z}_2\}$ forms a partition of $BF(n)$. With such observation, Wong [14] proposed a *stretching* operation to obtain $BF^i_\ell(n)$ from $BF(n-1)$. More precisely, the stretching operation can be described as follows. Assume that $\ell \in \mathbb{Z}_n$ and $i \in \mathbb{Z}_2$. Let $\Im_n$ be the set of all subgraphs of $BF(n)$ and let $G \in \Im_n$. We define the following subsets of $V(BF(n+1))$ and $E(BF(n+1))$:

$$\begin{aligned}
V_1 &= \{v^i_h \mid 0 \le h < \ell, v_h \in V(G)\},\\
V_2 &= \{v^i_{h+1} \mid \ell < h \le n-1, v_h \in V(G)\},\\
V_3 &= \{v^i_\ell \mid v_\ell \text{ is incident with}\\
&\qquad \text{a level-}(\ell-1)_{\bmod n} \text{ edge in } G\},\\
V_4 &= \{v^i_{\ell+1} \mid v_\ell \text{ is incident with}\\
&\qquad \text{a level-}\ell \text{ edge in } G\},\\
E_1 &= \{(v^i_h, v^i_{h+1}) \mid 0 \le h < \ell, (v_h, u_{h+1}) \in E(G)\},\\
E_2 &= \{(v^i_{h+1}, v^i_{h+2}) \mid \ell \le h \le n-1, (v_h, u_{h+1}) \in E(G)\},
\end{aligned}$$

and

$$\begin{aligned}
E_3 &= \{(v^i_\ell, v^i_{\ell+1}) \mid v_\ell \text{ is incident with at least one}\\
&\qquad \text{level-}(\ell-1)_{\bmod n} \text{ edge and at least one}\\
&\qquad \text{level-}\ell \text{ edge in } G\}
\end{aligned}$$

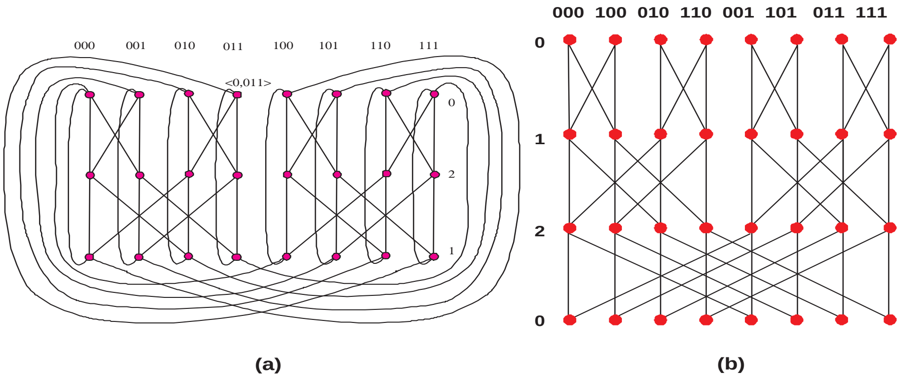


Figure 1: (a) The structure of $BF(3)$; (b) $BF(3)$ with level-0 replicated to ease visualization.

where

$$\begin{aligned} v_h &= \langle h, a_0 \dots a_{\ell-1} a_\ell \dots a_{n-1} \rangle, \\ u_h &= \langle h, b_0 \dots b_{\ell-1} b_\ell \dots b_{n-1} \rangle, \\ v_h^i &= \langle h, a_0 \dots a_{\ell-1} i a_\ell \dots a_{n-1} \rangle, \text{ and} \\ u_h^i &= \langle h, b_0 \dots b_{\ell-1} i b_\ell \dots b_{n-1} \rangle. \end{aligned}$$

Then we define the function $\gamma_\ell^i : \bigcup_{n \geq 3} \Im_n \to \bigcup_{n \geq 4} \Im_n$ by assigning $\gamma_\ell^i(G)$ as the graph with the vertex set $V_1 \cup V_2 \cup V_3 \cup V_4$ and the edge set $E_1 \cup E_2 \cup E_3$. One may find that γ_ℓ^i is well-defined and one-to-one. Furthermore, $\gamma_\ell^i(G) \in \Im_{n+1}$ if $G \in \Im_n$. In particular, $\gamma_\ell^i(BF(n)) = BF_\ell^i(n+1)$. Moreover, $\gamma_\ell^i(P)$ is a path in $BF(n+1)$ if P is a path in $BF(n)$.

In fact, $BF(n)$ can be further partitioned. Assume that $1 \leq m \leq n$, $i_1, \dots, i_m \in \mathbb{Z}_2$, and $\ell_1, \dots, \ell_m \in \mathbb{Z}_n$ such that $\ell_1 < \dots < \ell_m$. We use $BF_{\ell_1, \dots, \ell_m}^{i_1, \dots, i_m}(n)$ to denote the subgraph of $BF(n)$ induced by $\{\langle h, a_0 \dots a_{n-1} \rangle \mid h \in \mathbb{Z}_n, a_{\ell_j} = i_j \text{ for } 1 \leq j \leq m\}$. So, $\{BF_{\ell_1, \dots, \ell_m}^{i_1, \dots, i_m}(n) \mid i_1, \dots, i_m \in \mathbb{Z}_2, \ell_1, \dots, \ell_m \in \mathbb{Z}_n, \ell_1 < \dots < \ell_m\}$ forms a partition of $BF(n)$. To avoid the complicated case caused by modular arithmetic, we restrict our attention on $1 \leq m \leq n-1$, $0 \leq \ell_1 < \dots < \ell_m$, and $\ell_j < n - m + j - 1$ for each $1 \leq j \leq m$. Then the following lemmas can be easily derived.

Lemma 1. *Suppose that $i_1, \dots, i_m \in \mathbb{Z}_2$ and $\ell_1, \dots, \ell_m$ are integers such that $0 \leq \ell_1 < \dots < \ell_m$ and $\ell_j < n - m + j - 1$ for each $1 \leq j \leq m$. Then*

$$BF_{\ell_1, \dots, \ell_m}^{i_1, \dots, i_m}(n) = \begin{cases} \gamma_{\ell_m}^{i_m} \circ \dots \circ \gamma_{\ell_3}^{i_3}(BF_{\ell_1, \ell_2}^{i_1, i_2}(3)) & \text{if } m = n-1, \\ \gamma_{\ell_m}^{i_m} \circ \dots \circ \gamma_{\ell_2}^{i_2}(BF_{\ell_1}^{i_1}(3)) & \text{if } m = n-2, \\ \gamma_{\ell_m}^{i_m} \circ \dots \circ \gamma_{\ell_1}^{i_1}(BF(n-m)) & \text{otherwise.} \end{cases}$$

In the next lemma, we let

$$\begin{aligned} v &= \langle \ell, a_0 \dots a_{n-1} \rangle, \\ v_\ell^{ij} &= \langle \ell, a_0 \dots a_{\ell-1} ij a_\ell \dots a_{n-1} \rangle, \\ v_{\ell+1}^{ij} &= \langle \ell+1, a_0 \dots a_{\ell-1} ij a_\ell \dots a_{n-1} \rangle, \text{ and} \\ v_{\ell+2}^{ij} &= \langle \ell+2, a_0 \dots a_{\ell-1} ij a_\ell \dots a_{n-1} \rangle. \end{aligned}$$

Lemma 2. *Let $n \geq 3$. Assume that $0 \leq \ell \leq n-1$, $\Gamma \in \Im_n$, and G is a connected spanning subgraph of Γ. For any $i, j \in \mathbb{Z}_2$, let*

$$\begin{aligned} F_0 &= \{v \mid v \text{ is not incident with any level-}(\ell-1)_{\bmod n} \text{ edge in } G\}, \\ F_1 &= \{v \mid v \text{ is not incident with any level-}\ell \text{ edge in } G\}, \\ \overline{F_0} &= \bigcup_{v \in F_0} \{v_\ell^{ij}, v_{\ell+1}^{ij}\}, \\ \overline{F_1} &= \bigcup_{v \in F_1} \{v_{\ell+1}^{ij}, v_{\ell+2}^{ij}\}, \\ X_0 &= \bigcup_{v \in F_0} \{(v_\ell^{ij}, v_{\ell+1}^{ij})\}, \\ X_1 &= \bigcup_{v \in F_1} \{(v_{\ell+1}^{ij}, v_{\ell+2}^{ij})\}, \\ M_0 &= \bigcup_{v \in G - (F_0 \cup F_1)} \left\{(v_\ell^{ij}, v_{\ell+1}^{ij})\right\}, \text{ and} \\ M_1 &= \bigcup_{v \in G - (F_0 \cup F_1)} \{(v_{\ell+1}^{ij}, v_{\ell+2}^{ij})\}. \end{aligned}$$

Then $F_0 \cap F_1 = \phi$ and $\overline{F_0} \cap \overline{F_1} = \phi$. Thus, $\overline{F_0} \cup \overline{F_1} = V(\gamma_{\ell+1}^j \circ \gamma_\ell^i(\Gamma)) - V(\gamma_{\ell+1}^j \circ \gamma_\ell^i(G))$ can be represented as $\bigcup_{k=1}^m \{u_k, g(u_k) \mid u_k \in V(\gamma_{\ell+1}^j \circ \gamma_\ell^i(\Gamma))\}$ with some $m \geq 1$ such that $\{u_k \mid 1 \leq k \leq m\} \cap \{g(u_k) \mid 1 \leq k \leq m\} = \phi$. Moreover, $M_0 \cup M_1 \subseteq E(\gamma_{\ell+1}^j \circ \gamma_\ell^i(G))$ and $X_0 \cup X_1 = \bigcup_{k=1}^m \{(u_k, g(u_k)) \mid u_k \in V(\gamma_{\ell+1}^j \circ \gamma_\ell^i(\Gamma))\}$ is a set of edges with no shared endpoints.

Lemma 3. *Assume $n \geq 3$ and $k \geq 1$. Suppose that $\{P_1, \dots, P_k\}$ is a k-container of $BF(n)$ between two vertices x and y with the following conditions: (i) $V(BF(n)) - \bigcup_{i=1}^k V(P_i) = \bigcup_{i=1}^m \{u_i, g(u_i) \mid u_i \in V(BF(n))\}$ with some $m \geq 1$ such that $\{u_i \mid 1 \leq i \leq m\} \cap \{g(u_i) \mid 1 \leq i \leq m\} = \phi$, and (ii) $\bigcup_{i=1}^m \{(f(u_i), g^{-1} \circ f(u_i)) \subseteq \bigcup_{i=1}^k E(P_i)$. Then there exists a k^*-container of $BF(n)$ between x and y.*

Proof. Let $A = \bigcup_{i=1}^m \{(u_i, g(u_i))\} \cup \bigcup_{i=1}^m \{(u_i, f(u_i))\} \cup \bigcup_{i=1}^m \{(g(u_i), f^{-1} \circ g(u_i))\}$ and $B = \bigcup_{i=1}^m \{(f(u_i), g^{-1} \circ f(u_i))\}$. Obviously, $A \cap$

$\left(\bigcup_{i=1}^{k} E(P_i)\right) = \phi$. Then $\left(\left(\bigcup_{i=1}^{k} E(P_i)\right) \cup A\right) - B$ forms a k^*-container of $BF(n)$ between x and y. □

Let G be a subgraph of $BF(n)$ and let C be a cycle of G. Then C is an *ℓ-scheduled* cycle with respect to G if every level-ℓ vertex of G is incident with a level-$(\ell-1)_{\bmod n}$ edge and a level-ℓ edge in C. Furthermore, C is a *totally scheduled* cycle of G if it is an ℓ-scheduled cycle of G for all $\ell \in \mathbb{Z}_n$. Obviously, $\gamma_\ell^i(C)$ with $i \in \{0, 1\}$ is a totally scheduled cycle of $\gamma_\ell^i(G)$ if C is a totally scheduled cycle of G.

Theorem 1. *[14] Assume $n \geq 3$. Then $BF(n)$ is 1^*-connected if n is odd, and $BF(n)$ is 1^*-laceable otherwise.*

Theorem 2. *[14] Assume $n \geq 3$. Every $BF(n)$ has a totally scheduled hamiltonian cycle. Thus, $BF(n)$ is 2^*-connected*

By stretching operation, we have the following lemma and corollary.

Lemma 4. *Let $n \geq 3$. Assume that $0 \leq \ell \leq n-2$ and $i, j \in \mathbb{Z}_2$. Then there exists a totally scheduled hamiltonian cycle of $BF_{\ell,\ell+1}^{i,j}(n)$.*

Corollary 1. *Assume that $n \geq 4$ and $i, j, p, q \in \mathbb{Z}_2$. Then there exists a totally scheduled hamiltonian cycle of $BF_{0,1,2,3}^{i,j,p,q}(n)$, including all straight edges of level 0, level 1, level 2, and level 3 in $BF_{0,1,2,3}^{i,j,p,q}(n)$.*

3 Basic properties of $BF(n)$

A path P of $BF(n)$ is *ℓ-scheduled* if every level-ℓ vertex of $I(P)$ is incident with a level-$(\ell-1)_{\bmod n}$ edge and a level-ℓ edge on P. A path P of $BF(n)$ is *weakly ℓ-scheduled* if at least one level-ℓ vertex of $I(P)$ is incident with a level-$(\ell-1)_{\bmod n}$ edge and a level-ℓ edge on P.

Lemma 5. *Let $n \geq 3$. Assume $i, j \in \mathbb{Z}_2$ and $0 \leq \ell \leq n-3$. Suppose that s is any level-$(\ell+1)$ vertex of $BF_{\ell,\ell+1}^{i,j}(n)$ and d is any level-$(\ell+2)$ vertex of $BF_{\ell,\ell+1}^{i,j}(n)$. When $n = 3$, there exists a 0-scheduled hamiltonian path P_3 of $BF_{0,1}^{i,j}(3)$ joining s to d with $P_3(2) = g^{-1}(s)$. When $n \geq 4$, there exists an ℓ-scheduled hamiltonian path P_n of $BF_{\ell,\ell+1}^{i,j}(n)$ joining s to d such that P_n is weakly $(\ell+2)$-scheduled with $P_n(2) = g^{-1}(s)$. In particular, $P_n(n \times 2^{n-2} - 2) = g^{-2}(d)$ and $P_n(n \times 2^{n-2} - 1) = g^{-1}(d)$ for $n \geq 3$ if $d \neq g(s)$.*

Proof. Omitted. □

In terms of the symmetry of $BF(n)$, we have the following corollary.

Corollary 2. *Let $n \geq 3$. Assume $i, j \in \mathbb{Z}_2$ and $0 \leq \ell \leq n-3$. Suppose that s is any level-$(\ell+1)$ vertex of $BF_{\ell,\ell+1}^{i,j}(n)$ and d is any level-ℓ vertex of $BF_{\ell,\ell+1}^{i,j}(n)$. When $n = 3$, there exists a 2-scheduled hamiltonian path P_3 of $BF_{\ell,\ell+1}^{i,j}(n)$ joining s to d with $H(2) = g(s)$. When $n \geq 4$, there exists an $(\ell+2)$-scheduled hamiltonian path P_n of $BF_{\ell,\ell+1}^{i,j}(n)$ joining s to d such that P_n is weakly ℓ-scheduled with $P_n(2) = g(s)$. In particular, $P_n(n \times 2^{n-2} - 2) = g^2(d)$ and $P_n(n \times 2^{n-2} - 1) = g(d)$ for $n \geq 3$ if $d \neq g^{-1}(s)$.*

Lemma 6. *Assume that $n \geq 3$. For any $x \in \mathbb{Z}_2^n$, let $F_x = \{\langle h, x\rangle \mid h \in \mathbb{Z}_n\}$. Then there is a totally scheduled hamiltonian cycle in $BF(n) - F_x$.*

Proof. Without loss of generality, we assume $x = 0^n$. Then we prove this lemma by induction on n. The induction basis is a totally scheduled hamiltonian cycle of $BF(3) - \{\langle 0, 000\rangle, \langle 1, 000\rangle, \langle 2, 000\rangle\}$, listed in Table 1.

Now we suppose that the statement holds for $BF(n-1)$ with $n \geq 4$ and partition $BF(n)$ into $\{BF_0^0(n), BF_0^1(n)\}$. Thus, $F_x \subset V(BF_0^0(n))$. Let $F'_x = \{\langle h, 0^{n-1}\rangle \mid h \in \mathbb{Z}_{n-1}\}$. By induction hypothesis, there exists a totally scheduled hamiltonian cycle C_0 of $BF(n-1) - F'_x$. By Theorem 2, there exists a totally scheduled cycle C_1 of $BF(n-1)$. Since $BF_0^0(n) - F_x = \gamma_0^0(BF(n-1) - F'_x)$ and $BF_0^1(n) = \gamma_0^1(BF(n-1))$, then $\gamma_0^0(C_0)$ and $\gamma_0^1(C_1)$ are totally scheduled hamiltonian cycles of $BF_0^0(n) - F_x$ and $BF_0^1(n)$, respectively. Let C be the subgraph of $BF(n)$ generated by $E(\gamma_0^0(C_0)) \cup E(\gamma_0^1(C_1)) \cup \{(\langle 0, 01^{n-1}\rangle, \langle 1, 1^n\rangle), (\langle 0, 1^n\rangle, \langle 1, 01^{n-1}\rangle)\} - \{(\langle 0, 01^{n-1}\rangle, \langle 1, 01^{n-1}\rangle), (\langle 0, 1^n\rangle, \langle 1, 1^n\rangle)\}$. Then C is a totally scheduled hamiltonian cycle of $BF(n) - F_x$. □

Lemma 7. *Assume $i, j \in \mathbb{Z}_2$, n is an odd integer greater than or equal to 3, and $x \in \mathbb{Z}_2^{n-2}$. Let $u_h = \langle h, ijx\rangle$ for any $h \in \mathbb{Z}_n$. Then the following statements are true: (i) There exists a hamiltonian cycle in $BF_{0,1}^{i,j}(n) - \{u_1\}$; (ii) there exists a 0-scheduled hamiltonian cycle in $BF_{0,1}^{i,j}(n) - \{u_0, u_1, u_{n-1}\}$; (iii) there exists a 0-scheduled hamiltonian cycle in $BF_{0,1}^{i,j}(n) - \{u_0, u_1, u_2\}$.*

Proof. By Lemma 6, there is a totally scheduled hamiltonian cycle C_f in $BF(n) - \{u_h \mid h \in \mathbb{Z}_n\}$. With regard to each statement, we construct a cycle as follows:

Let C_1 be the subgraph of $BF_{0,1}^{i,j}(n)$ generated by $(E(C_f) \cup \{(u_{2t}, u_{(2t+1)_{\bmod n}}) \mid 1 \leq t \leq \lfloor \frac{n}{2} \rfloor\} \cup \{(u_{2t}, f(u_{2t})) \mid 1 \leq t \leq \lfloor \frac{n}{2} \rfloor\} \cup \{(u_{(2t+1)_{\bmod n}}, f^{-1}(u_{(2t+1)_{\bmod n}})) \mid 1 \leq t \leq \lfloor \frac{n}{2} \rfloor\}) - \{(f(u_{2t}), f^{-1}(u_{(2t+1)_{\bmod n}})) \mid 1 \leq t \leq \lfloor \frac{n}{2} \rfloor\}$. Then C_1 is a hamiltonian cycle in $BF_{0,1}^{i,j}(n) - \{u_1\}$.

Let C_2 be the subgraph of $BF_{0,1}^{i,j}(n)$ generated by $(E(C_f) \cup \{(u_{2t}, u_{2t+1}) \mid 1 \leq t \leq \lfloor \frac{n}{2} \rfloor - 1\} \cup \{(u_{2t}, f(u_{2t})) \mid 1 \leq t \leq \lfloor \frac{n}{2} \rfloor - 1\} \cup \{(u_{2t+1}, f^{-1}(u_{2t+1})) \mid 1 \leq t \leq \lfloor \frac{n}{2} \rfloor - 1\}) - \{(f(u_{2t}), f^{-1}(u_{2t+1})) \mid 1 \leq t \leq \lfloor \frac{n}{2} \rfloor - 1\}$. Then C_2 forms a 0-scheduled hamiltonian cycle in $BF_{0,1}^{i,j}(n) - \{u_0, u_1, u_{n-1}\}$.

Let C_3 be the subgraph of $BF_{0,1}^{i,j}(n)$ generated by $(E(C_f) \cup \{(u_{2t-1}, u_{2t}) \mid 2 \leq t \leq \lfloor \frac{n}{2} \rfloor\} \cup$

Table 1: A totally scheduled hamiltonian cycle of $BF(3) - \{\langle 0,000\rangle, \langle 1,000\rangle, \langle 2,000\rangle\}$ as induction basis.

$\langle\langle 0,100\rangle, \langle 1,100\rangle, \langle 2,110\rangle, \langle 0,111\rangle, \langle 1,011\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,111\rangle, \langle 2,101\rangle, \langle 0,101\rangle, \langle 1,001\rangle,$ $\langle 2,001\rangle, \langle 0,001\rangle, \langle 1,101\rangle, \langle 2,111\rangle, \langle 0,110\rangle, \langle 1,010\rangle, \langle 2,010\rangle, \langle 0,010\rangle, \langle 1,110\rangle, \langle 2,100\rangle, \langle 0,100\rangle\rangle$

$\{(u_{2t-1}, f(u_{2t-1})) \mid 2 \leq t \leq \lfloor \frac{n}{2} \rfloor\} \cup \{(u_{2t}, f^{-1}(u_{2t})) \mid 2 \leq t \leq \lfloor \frac{n}{2} \rfloor\}) - \{(f(u_{2t-1}), f^{-1}(u_{2t})) \mid 2 \leq t \leq \lfloor \frac{n}{2} \rfloor\}$. Then C_3 is a 0-scheduled hamiltonian cycle in $BF_{0,1}^{i,j}(n) - \{u_0, u_1, u_2\}$. □

Lemma 8. *Assume that n is an odd integer greater than or equal to 3. Let s and d be two distinct level-0 vertices of $BF(n)$. Then there exists a hamiltonian path of $BF(n)$ joining s to d such that s is incident with a level-$(n-1)$ edge and d is incident with a level-0 edge.*

Proof. Without loss of generality, we assume $s = \langle 0, 0^n\rangle$ and $d = \langle 0, ijx\rangle$ with some $i, j \in \mathbb{Z}_2$ and $x \in \mathbb{Z}_2^{n-2}$. Then we construct the desired hamiltonian path of $BF(n)$ by induction on n. The induction bases depend upon the hamiltonian paths of $BF(3)$ joining $\langle 0,000\rangle$ to the other level-0 vertices, enumerated in Table 2. Then we suppose that the statement holds for $BF(n-2)$ with $n \geq 5$ and partition $BF(n)$ into $\{BF_{0,1}^{p,q}(n) \mid p, q \in \mathbb{Z}_2\}$.

Case 1: Suppose that $x = 0^{n-2}$. Let $t = \langle 0, y\rangle$ be a level-0 vertex of $BF(n-2)$ other than $s' = \langle 0, 0^{n-2}\rangle$. By induction hypothesis, there is a hamiltonian path Q of $BF(n-2)$ joining s' to t such that s' is incident with a level-$(n-3)$ edge and t is incident with a level-0 edge. Furthermore, let $u_h^{pq} = \langle h, pq0^{n-2}\rangle$ and $t_h^{pq} = \langle h, pqy\rangle$ for any $p, q \in \mathbb{Z}_2$ and $h \in \{0,1,2\}$. Since $BF_{0,1}^{0,0}(n) = \gamma_1^0 \circ \gamma_0^0(BF(n-2))$, $\gamma_1^0 \circ \gamma_0^0(Q)$ is a path of $BF_{0,1}^{0,0}(n)$ joining $s = u_0^{00}$ to t_2^{00}. By Corollary 2, there is a 2-scheduled hamiltonian path $H^{ij} = \langle t_1^{ij}, P_{ij}, u_2^{ij}, u_1^{ij}, u_0^{ij}\rangle$ of $BF_{0,1}^{i,j}(n)$ joining t_1^{ij} to u_0^{ij}. By Lemma 4, there is a totally scheduled hamiltonian cycle $C^{pq} = \langle t_0^{pq}, t_1^{pq}, t_2^{pq}, D_{pq}, u_0^{pq}, u_1^{pq}, u_2^{pq}, R_{pq}, t_0^{pq}\rangle$ of $BF_{0,1}^{p,q}(n)$ for $pq \in \mathbb{Z}_2^2 - \{00, ij\}$.

Subcase 1.1: If $ij = 10$, $d = u_0^{10}$. Then $J = \langle$ s, $\gamma_1^0 \circ \gamma_0^0(Q)$, t_2^{00}, t_1^{00}, t_0^{00}, t_1^{10}, P_{10}, u_2^{10}, u_1^{11}, u_0^{01}, D_{01}^{-1}, t_2^{01}, t_1^{01}, t_0^{01}, R_{01}^{-1}, u_2^{01}, u_1^{01}, u_0^{11}, D_{11}^{-1}, t_2^{11}, t_1^{11}, t_0^{11}, R_{11}^{-1}, u_2^{11}, u_1^{10},d $\rangle$ is a path of $BF(n)$ joining s to d. See Figure 2(a) for illustration.

Subcase 1.2: If $ij = 01$, $d = u_0^{01}$. Then $J = \langle$ s, $\gamma_1^0 \circ \gamma_0^0(Q)$, t_2^{00}, t_1^{01}, P_{01}, u_2^{01}, u_1^{01}, u_0^{11}, D_{11}^{-1}, t_2^{11}, t_1^{11}, t_0^{11}, R_{11}^{-1}, u_2^{11}, u_1^{10}, u_0^{10}, D_{10}^{-1}, t_2^{10}, t_1^{10}, t_0^{10}, R_{10}^{-1}, u_2^{10}, u_1^{11}, d $\rangle$ joins s to d. See Figure 2(b).

Subcase 1.3: If $ij = 11$, $d = u_0^{11}$. Then $J = \langle$ s, $\gamma_1^0 \circ \gamma_0^0(Q)$, t_2^{00}, t_1^{01}, t_2^{01}, D_{01}, u_0^{01}, u_1^{01}, u_2^{01}, R_{01}, t_0^{01}, t_1^{11}, P_{11}, u_2^{11}, u_1^{10}, u_0^{10}, D_{10}^{-1}, t_2^{10}, t_1^{10}, t_0^{10}, R_{10}^{-1}, u_0^{10}, u_1^{11}, $d\rangle$ joins s to d. See Figure 2(c).

Case 2: Suppose that $x \neq 0^{n-2}$. By induction hypothesis, there is a hamiltonian path Q of $BF(n-2)$ joining $s' = \langle 0, 0^{n-2}\rangle$ to $d' = \langle 0, x\rangle$ such that s' is incident with a level-$(n-3)$ edge and d' is incident with a level-0 edge. Furthermore, let $s_h^{pq} = \langle h, pq0^{n-2}\rangle$ and $d_h^{pq} = \langle h, pqx\rangle$ for any $p, q \in \mathbb{Z}_2$, $h \in \{0,1,2\}$. Obviously, $\gamma_1^0 \circ \gamma_0^0(Q)$ joins $s_0^{00} = s$ to d_2^{00}. By Lemma 4, there is a totally scheduled hamiltonian cycle $C^{pq} = \langle d_0^{pq}, d_1^{pq}, d_2^{pq}, R_{pq}, d_0^{pq}\rangle$ of $BF_{0,1}^{p,q}(n)$ for any $p, q \in \mathbb{Z}_2$. By Corollary 2, there is a hamiltonian path $T^{ij} = \langle s_1^{ij}, W_{ij}, d_2^{ij}, d_1^{ij}, d\rangle$ of $BF_{0,1}^{i,j}(n)$ joining s_1^{ij} to d.

Subcase 2.1: If $ij = 00$, $d = d_0^{00}$. Then $J = \langle s$, $\gamma_1^0 \circ \gamma_0^0(Q)$, d_2^{00}, d_1^{00}, d_0^{10}, R_{10}^{-1}, d_2^{10}, d_1^{11}, d_0^{01}, R_{01}^{-1}, d_2^{01}, d_1^{01}, d_0^{11}, R_{11}^{-1}, d_2^{11}, d_1^{10}, $d\rangle$ joins s to d. See Figure 3(a).

Subcase 2.2: If $ij = 10$, $d = d_0^{10}$. By Lemma 5, there is a hamiltonian path $L^{11} = \langle$ d_1^{11}, D_{11}, s_0^{11}, s_1^{11}, $s_2^{11}\rangle$ of $BF_{0,1}^{1,1}(n)$ joining d_1^{11} to s_2^{11}. Then $J = \langle s$, $\gamma_1^0 \circ \gamma_0^0(Q)$, d_2^{00}, d_1^{01}, d_2^{01}, R_{01}, d_0^{01}, d_1^{11}, D_{11}, s_0^{11}, s_1^{11}, s_2^{11}, s_1^{10}, W_{10}, d_2^{10}, d_1^{10}, $d\rangle$ joins s to d. See Figure 3(b).

Subcase 2.3: If $ij = 01$, $d = d_0^{01}$. By Corollary 2, there is a hamiltonian path $S^{11} = \langle$ d_1^{11}, U_{11}, s_2^{11}, s_1^{11}, s_0^{11} $\rangle$ of $BF_{0,1}^{1,1}(n)$. Then $J = \langle s$, $\gamma_1^0 \circ \gamma_0^0(Q)$, d_2^{00}, d_1^{00}, d_0^{00}, d_1^{10}, d_0^{10}, R_{10}^{-1}, d_2^{10}, d_1^{11}, U_{11}, s_2^{11}, s_1^{11}, s_0^{11}, s_1^{01}, W_{01}, d_2^{01}, d_1^{01}, $d\rangle$ joins s to d. See Figure 3(c) for illustration.

Subcase 2.4: If $ij = 11$, $d = d_0^{11}$. By Corollary 2, there is a hamiltonian path $S^{01} = \langle$ d_1^{01}, U_{01}, s_2^{01}, s_1^{01}, s_0^{01} $\rangle$ of $BF_{0,1}^{0,1}(n)$. Then $J = \langle s$, $\gamma_1^0 \circ \gamma_0^0(Q)$, d_2^{00}, d_1^{01}, U_{01}, s_2^{01}, s_1^{01}, s_0^{01}, s_1^{11}, W_{11}, d_2^{11}, d_1^{10}, d_0^{10}, R_{10}^{-1}, d_2^{10}, d_1^{11}, $d\rangle$ joins s to d. See Figure 3(d).

Note that J is a 1-container of $BF(n)$ joining s to d. By Lemma 2, $V(BF(n)) - V(J) = V(BF_{0,1}^{0,0}(n)) - V(\gamma_1^0 \circ \gamma_0^0(Q)) = \bigcup_{i=1}^m \{u_i, g(u_i)\}$ for some $m \geq 1$ with $\{u_i \mid 1 \leq i \leq m\} \cap \{g(u_i) \mid 1 \leq i \leq m\} = \phi$. Furthermore, $\bigcup_{i=1}^m \{(f(u_i), g^{-1} \circ f(u_i))\} \subseteq E(J)$. By Lemma 3, there exists a 1^*-container of $BF(n)$ joining s to d. Thus, the requirements are all satisfied. □

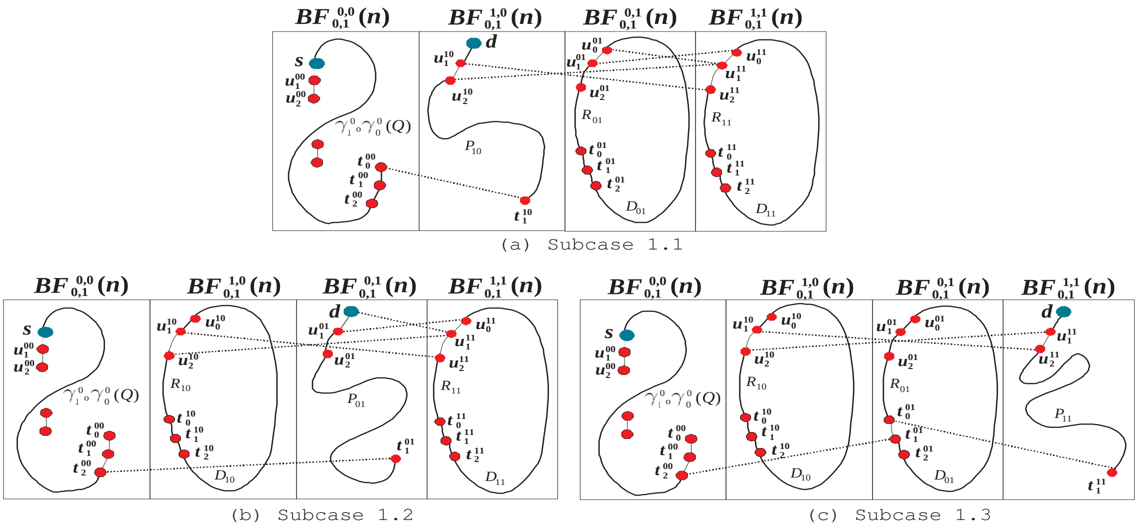
4 3^*-containers of $BF(n)$

Based on the symmetry of $BF(n)$, only the containers between two vertices at the same level and the containers between two vertices of odd level differences are concerned.

Assume that $\ell \in \mathbb{Z}_n$ for $n \geq 3$. A subgraph G of $BF(n)$ is *ℓ-designed* if G spans $BF(n)$ and every level-ℓ vertex of G is incident with at least one level-$(\ell-1)_{\bmod n}$ edge and at least one level-ℓ edge. Obviously, $\gamma_{\ell+1}^j \circ \gamma_\ell^i(G)$ spans $\gamma_{\ell+1}^j \circ \gamma_\ell^i(BF(n)) = BF_{\ell,\ell+1}^{i,j}(n+2)$ if G is ℓ-designed. Let $s = \langle \ell, a_0 a_1 \ldots a_{n-1}\rangle$ and $d = \langle \ell, b_0 b_1 \ldots b_{n-1}\rangle$ be two distinct level-ℓ vertices of $BF(n)$. Since $BF(n)$ is vertex-transitive, we define the automorphism $\mu_{s,d} = \mu_{n-1} \circ \ldots \circ \mu_1 \circ \mu_0$ over $V(BF(n))$ where for $0 \leq i \leq n-1$, $\mu_i = g$ if $a_{(i+\ell)_{\bmod n}} = b_{(i+\ell)_{\bmod n}}$, and $\mu_i = f$

Table 2: Hamiltonian paths of $BF(3)$ as the induction bases.

d	A hamiltonian path of $BF(3)$ from ⟨0, 000⟩ to d with the desired property
⟨0, 100⟩	⟨⟨0, 000⟩, ⟨2, 001⟩, ⟨1, 001⟩, ⟨0, 101⟩, ⟨2, 101⟩, ⟨1, 111⟩, ⟨0, 011⟩, ⟨2, 011⟩, ⟨1, 011⟩, ⟨0, 111⟩, ⟨2, 111⟩, ⟨1, 101⟩, ⟨0, 001⟩, ⟨2, 000⟩, ⟨1, 010⟩, ⟨0, 110⟩, ⟨2, 110⟩, ⟨1, 100⟩, ⟨2, 100⟩, ⟨1, 110⟩, ⟨0, 010⟩, ⟨2, 010⟩, ⟨1, 000⟩, ⟨0, 100⟩⟩
⟨0, 010⟩	⟨⟨0, 000⟩, ⟨2, 001⟩, ⟨1, 001⟩, ⟨0, 101⟩, ⟨2, 101⟩, ⟨1, 111⟩, ⟨0, 011⟩, ⟨2, 011⟩, ⟨1, 011⟩, ⟨0, 111⟩, ⟨2, 111⟩, ⟨1, 101⟩, ⟨0, 001⟩, ⟨2, 000⟩, ⟨1, 000⟩, ⟨2, 010⟩, ⟨1, 010⟩, ⟨0, 110⟩, ⟨2, 110⟩, ⟨1, 100⟩, ⟨0, 100⟩, ⟨2, 100⟩, ⟨1, 110⟩, ⟨0, 010⟩⟩
⟨0, 110⟩	⟨⟨0, 000⟩, ⟨2, 001⟩, ⟨1, 001⟩, ⟨0, 101⟩, ⟨2, 101⟩, ⟨1, 111⟩, ⟨0, 011⟩, ⟨2, 011⟩, ⟨1, 011⟩, ⟨0, 111⟩, ⟨2, 111⟩, ⟨1, 101⟩, ⟨0, 001⟩, ⟨2, 000⟩, ⟨1, 000⟩, ⟨0, 100⟩, ⟨2, 100⟩, ⟨1, 100⟩, ⟨2, 110⟩, ⟨1, 110⟩, ⟨0, 010⟩, ⟨2, 010⟩, ⟨1, 010⟩, ⟨0, 110⟩⟩
⟨0, 001⟩	⟨⟨0, 000⟩, ⟨2, 001⟩, ⟨1, 001⟩, ⟨0, 101⟩, ⟨2, 100⟩, ⟨1, 110⟩, ⟨0, 010⟩, ⟨2, 010⟩, ⟨1, 000⟩, ⟨2, 000⟩, ⟨1, 010⟩, ⟨0, 110⟩, ⟨2, 110⟩, ⟨1, 100⟩, ⟨0, 100⟩, ⟨2, 101⟩, ⟨1, 111⟩, ⟨0, 011⟩, ⟨2, 011⟩, ⟨1, 011⟩, ⟨0, 111⟩, ⟨2, 111⟩, ⟨1, 101⟩, ⟨0, 001⟩⟩
⟨0, 101⟩	⟨⟨0, 000⟩, ⟨2, 001⟩, ⟨1, 001⟩, ⟨0, 001⟩, ⟨2, 000⟩, ⟨1, 000⟩, ⟨2, 010⟩, ⟨1, 010⟩, ⟨0, 110⟩, ⟨2, 110⟩, ⟨1, 100⟩, ⟨0, 100⟩, ⟨2, 100⟩, ⟨1, 110⟩, ⟨0, 010⟩, ⟨2, 011⟩, ⟨0, 011⟩, ⟨1, 011⟩, ⟨0, 111⟩, ⟨2, 111⟩, ⟨1, 111⟩, ⟨2, 101⟩, ⟨1, 101⟩, ⟨0, 101⟩⟩
⟨0, 011⟩	⟨⟨0, 000⟩, ⟨2, 001⟩, ⟨0, 001⟩, ⟨1, 101⟩, ⟨2, 101⟩, ⟨0, 100⟩, ⟨1, 100⟩, ⟨2, 110⟩, ⟨0, 110⟩, ⟨1, 010⟩, ⟨2, 000⟩, ⟨1, 000⟩, ⟨2, 010⟩, ⟨0, 010⟩, ⟨1, 110⟩, ⟨2, 100⟩, ⟨0, 101⟩, ⟨1, 001⟩, ⟨2, 011⟩, ⟨1, 011⟩, ⟨0, 111⟩, ⟨2, 111⟩, ⟨1, 111⟩, ⟨0, 011⟩⟩
⟨0, 111⟩	⟨⟨0, 000⟩, ⟨2, 001⟩, ⟨0, 001⟩, ⟨1, 101⟩, ⟨2, 111⟩, ⟨1, 111⟩, ⟨2, 101⟩, ⟨0, 100⟩, ⟨1, 100⟩, ⟨2, 110⟩, ⟨0, 110⟩, ⟨1, 010⟩, ⟨2, 000⟩, ⟨1, 000⟩, ⟨2, 010⟩, ⟨0, 010⟩, ⟨1, 110⟩, ⟨2, 100⟩, ⟨0, 101⟩, ⟨1, 001⟩, ⟨2, 011⟩, ⟨0, 011⟩, ⟨1, 011⟩, ⟨0, 111⟩⟩

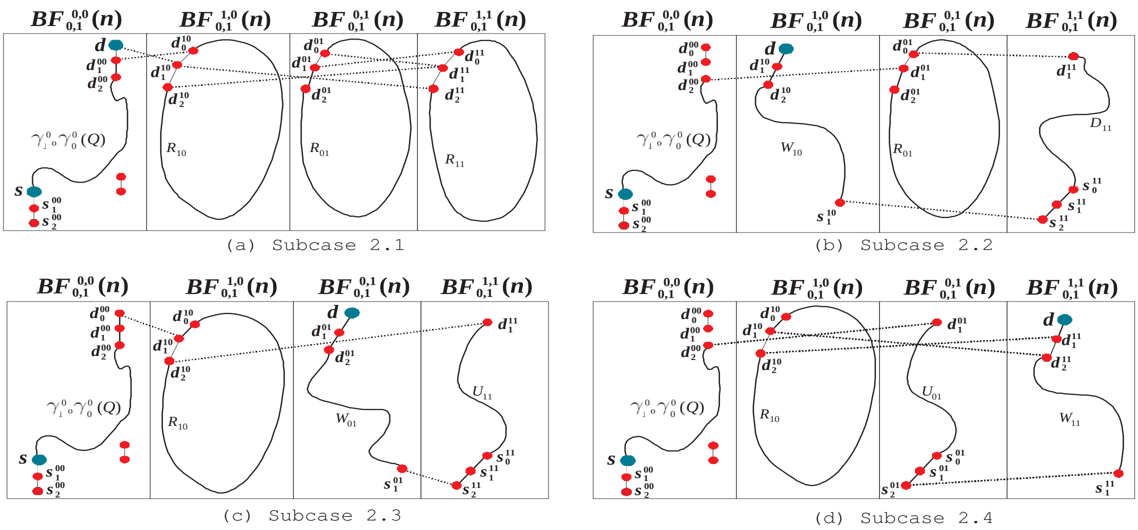


(a) Subcase 1.1

(b) Subcase 1.2

(c) Subcase 1.3

Figure 2: Illustrations for Case 1 of Lemma 8.



(a) Subcase 2.1

(b) Subcase 2.2

(c) Subcase 2.3

(d) Subcase 2.4

Figure 3: Illustrations for Case 2 of Lemma8.

otherwise. For any path joining s to d, say $P = \langle s = v_0, v_1, ..., v_k = d\rangle$ with some $k \geq 1$, we further define $\mu_{s,d}(P) = \langle d = \mu_{s,d}(v_0), \mu_{s,d}(v_1), ..., \mu_{s,d}(v_k) = s\rangle$.

For the sake of clarity, the proofs of the following lemmas are described in Appendix.

Lemma 9. *Assume that n is an odd integer greater than or equal to 3 and also that $\ell \in \mathbb{Z}_n$. Let s and d be two distinct vertices at level ℓ of $BF(n)$. Then there exists a 3^*-container $\{P_1, P_2, P_3\}$ of $BF(n)$ joining s to d such that the following requirements are all satisfied: (i) both P_2 and P_3 begin with level-ℓ edges, (ii) only one path of $\{P_1, P_2, P_3\}$ ends up with a level-ℓ edge, (iii) at least one of P_2 and P_3 is weakly ℓ-scheduled, (iv) there are two vertices v_1, v_2 of $I(P_2) \cup I(P_3)$ so that $\mu_{s,d}(v_1) = v_2$ and each of $\{v_1, v_2\}$ is incident with a level-$(\ell-1)_{mod\ n}$ edge and a level-ℓ edge, and (v) the subgraph generated by $E(P_1) \cup E(P_2) \cup E(P_3)$ is ℓ-designed.*

Lemma 10. *For $n \geq 3$, assume that ℓ_s and ℓ_d are two integers of $\mathbb{Z}_n$ such that $\ell_d - \ell_s = 1$. Let s be any level-ℓ_s vertex of $BF(n)$ and d be any level-ℓ_d vertex of $BF(n)$. Then there exists a 3^*-container $\{P_1, P_2, P_3\}$ of $BF(n)$ joining s to d such that the following requirements are satisfied: (i) only P_1 ends up with a level-ℓ_d edge, (ii) $(s, g(s)) \in E(P_2) \cup E(P_3)$, (iii) at least one of P_2 and P_3 is weakly ℓ_d-scheduled, and (iv) the subgraph generated by $E(P_1) \cup E(P_2) \cup E(P_3)$ is ℓ_d-designed.*

Lemma 11. *Let s be any level-0 vertex of $BF(4)$ and d be any level-3 vertex of $BF(4)$. Then there exists a 3^*-container $\{P_1, P_2, P_3\}$ of $BF(4)$ joining s to d such that the following requirements are satisfied: (i) only P_1 ends up with a level-3 edge, (ii) P_2 or P_3 is weakly 3-scheduled, and (iii) the subgraph generated by $E(P_1) \cup E(P_2) \cup E(P_3)$ is 3-designed.*

Lemma 12. *For $n \geq 5$, assume that ℓ_s and ℓ_d are two integers of $\mathbb{Z}_n$ such that $\ell_d - \ell_s$ is odd between 3 and $n-1$. Let s be any level-ℓ_s vertex of $BF(n)$ and d be any level-ℓ_d vertex of $BF(n)$. Then there exists a 3^*-container $\{P_1, P_2, P_3\}$ of $BF(n)$ joining s to d such that the following requirements are satisfied: (i) only P_1 ends up with a level-ℓ_d edge, (ii) at least one of P_2 and P_3 is weakly ℓ_d-scheduled, and (iii) the subgraph generated by $E(P_1) \cup E(P_2) \cup E(P_3)$ is ℓ_d-designed.*

By Lemma 9, Lemma 10, Lemma 11, and Lemma 12, we derive the following result.

Theorem 3. *Let $n \geq 3$. Then $BF(n)$ is 3^*-connected if n is odd and is 3^*-laceable otherwise.*

5 4^*-containers of $BF(n)$

By Lemma 9 and the automorphism $\mu_{s,d}$ defined above, we have the following corollary.

Corollary 3. *Assume that n is an odd integer greater than or equal to 3 and also that $\ell \in \mathbb{Z}_n$. Let s and d be two distinct level-ℓ vertices of $BF(n)$. By Lemma 9, there is a 3^*-containers $\{P_1, P_2, P_3\}$ where P_1 begins with a level-$(\ell-1)_{mod\ n}$ edge. Then $\Omega = \{Q_1 = (\mu_{s,d}(P_1))^{-1}, Q_2 = (\mu_{s,d}(P_2))^{-1}, Q_3 = (\mu_{s,d}(P_3))^{-1}\}$ is also a 3^*-container of $BF(n)$ joining s to d with the following conditions: (i) only one path of Ω begins with a level-ℓ edge, (ii) only Q_1 ends up with a level-$(\ell-1)_{mod\ n}$ edge, (iii) at least one path of $\{Q_2, Q_3\}$ is weakly ℓ-scheduled, (iv) there are two level-ℓ vertices v_1, v_2 of $(I(P_2) \cup I(P_3)) \cap (I(Q_2) \cup I(Q_3))$ such that $\mu_{s,d}(v_1) = v_2$ and each of v_1 and v_2 is incident with a level-$(\ell-1)_{mod\ n}$ edge and a level-ℓ edge, and (v) the subgraph generated by $E(Q_1) \cup E(Q_2) \cup E(Q_3)$ is ℓ-designed.*

With Lemma 9 and Corollary 3, we have the following proposition. The proof is described in Appendix.

Proposition 1. *Assume that n is an odd integer greater than or equal to 3 and also that $\ell \in \mathbb{Z}_n$. Let s and d be two distinct level-ℓ vertices of $BF(n)$. Then there is a 3^*-container of $BF(n)$ between s and d such that each of $\{s, d\}$ is incident with only one level-ℓ edge.*

Lemma 13. *Assume that n is an odd integer greater than or equal to 3. Let s and d be two distinct vertices at the same level of $BF(n)$. Then there exists a 4^*-container of $BF(n)$ between s and d.*

Proof. Without loss of generality, we assume that $s = \langle 0, 0^n\rangle$ and $d = \langle 0, ijx\rangle$ with some $i, j \in \mathbb{Z}_2$ and $x \in \mathbb{Z}_2^{n-2}$. The desired 4^*-containers of $BF(3)$ are enumerated in Table 3. When $n \geq 5$, $BF(n)$ is partitioned into $\{BF_{0,1}^{p,q}(n) \mid p, q \in \mathbb{Z}_2\}$.

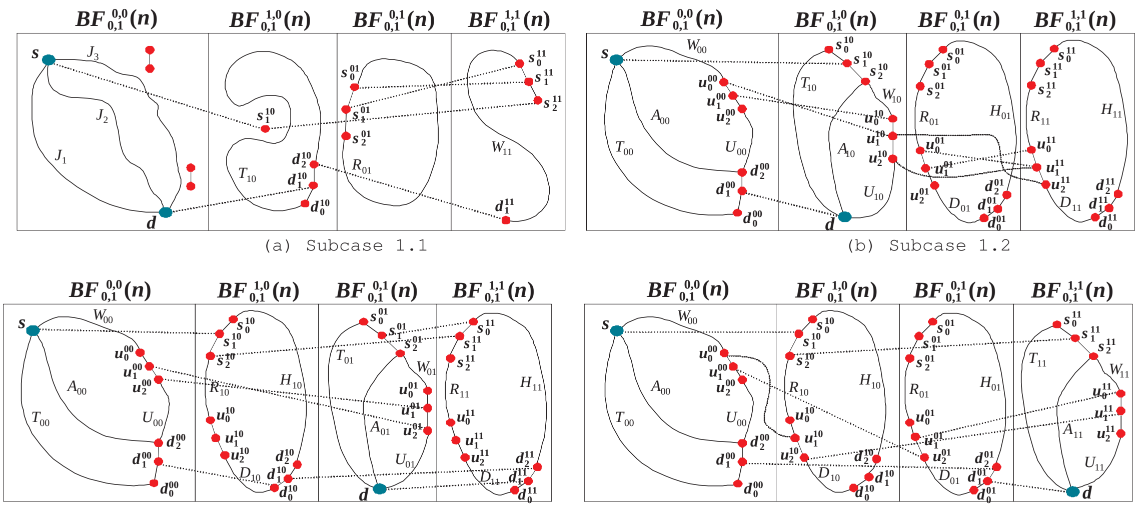
Case 1: Suppose that $x \neq 0^{n-2}$. Let $s_h^{pq} = \langle h, pq0^{n-2}\rangle$ for any $p, q \in \mathbb{Z}_2$ and $h \in \{0, 1, 2\}$. Then we have to consider the following subcases.

Subcase 1.1: Assume that $ij = 00$. By Proposition 1, we build a 3^*-container $\{P_1, P_2, P_3\}$ of $BF(n-2)$ joining $s' = \langle 0, 0^{n-2}\rangle$ to $d' = \langle 0, x\rangle$. Let Γ be the subgraph of $BF(n-2)$ generated by $E(P_1) \cup E(P_2) \cup E(P_3)$. Since $BF_{0,1}^{0,0}(n) = \gamma_1^0 \circ \gamma_0^0(BF(n-2))$, $\gamma_1^0 \circ \gamma_0^0(\Gamma)$ consists of three disjoint paths $\{J_1, J_2, J_3\}$ of $BF_{0,1}^{0,0}(n)$ joining s to d. By Lemma 4, there is a totally scheduled hamiltonian cycle $C^{01} = \langle s_0^{01}, s_1^{01}, s_2^{01}, R_{01}, s_0^{01}\rangle$ of $BF_{0,1}^{0,1}(n)$. By Lemma 7, there is a hamiltonian cycle $C^{10} = \langle d_0^{10}, d_1^{10}, d_2^{10}, T_{10}, d_0^{10}\rangle$ of $BF_{0,1}^{1,0}(n) - \{s_1^{10}\}$. By Lemma 5, there is a hamiltonian path $P^{11} = \langle s_2^{11}, s_1^{11}, s_0^{11}, W_{11}, d_1^{11}\rangle$ of $BF_{0,1}^{1,1}(n)$. Then, let $J_4 = \langle s$, s_1^{10}, s_2^{11}, s_1^{11}, s_0^{01}, R_{01}^{-1}, s_2^{01}, s_1^{01}, s_0^{11}, W_{11}, d_1^{11}, d_2^{10}, T_{10}, d_0^{10}, d_1^{10}, $d\rangle$, and thus $\{J_1, J_2, J_3, J_4\}$ is a 4-container of $BF(n)$ joining s to d. See Figure 4(a) for illustration.

By Lemma 2, $V(BF(n)) - \bigcup_{i=1}^4 V(J_i) = V(BF_{0,1}^{0,0}(n)) - V(\gamma_1^0 \circ \gamma_0^0(\Gamma)) = \bigcup_{j=1}^m \{u_j, g(u_j)\}$ for some $m \geq 1$ with $\{u_j \mid 1 \leq j \leq m\} \cap \{g(u_j) \mid 1 \leq j \leq m\} = \phi$. Moreover, $\bigcup_{j=1}^m \{(f(u_j), g^{-1} \circ f(u_j))\} \subseteq \bigcup_{i=1}^4 E(J_i)$. Thus, by Lemma 3, there is a 4^*-container of $BF(n)$ joining s to d.

Table 3: 4^*-containers of $BF(3)$ joining $\langle 0, 000\rangle$ to the other level-0 vertices.

d	4^*-container $\{J_1, J_2, J_3, J_4\}$ joining $\langle 0, 000\rangle$ to d.
$\langle 0, 100\rangle$	$J_1 = \langle\langle 0, 000\rangle, \langle 1, 000\rangle, \langle 0, 100\rangle\rangle$
	$J_2 = \langle\langle 0, 000\rangle, \langle 1, 100\rangle, \langle 0, 100\rangle\rangle$
	$J_3 = \langle\langle 0, 000\rangle, \langle 2, 000\rangle, \langle 1, 010\rangle, \langle 2, 010\rangle, \langle 0, 010\rangle, \langle 1, 110\rangle, \langle 2, 100\rangle, \langle 0, 100\rangle\rangle$
	$J_4 = \langle\langle 0, 000\rangle, \langle 2, 001\rangle, \langle 0, 001\rangle, \langle 1, 001\rangle, \langle 0, 101\rangle, \langle 1, 101\rangle, \langle 2, 111\rangle, \langle 0, 110\rangle, \langle 2, 110\rangle, \langle 0, 111\rangle, \langle 1, 011\rangle, \langle 2, 011\rangle, \langle 0, 011\rangle, \langle 1, 111\rangle, \langle 2, 101\rangle, \langle 0, 100\rangle\rangle$
$\langle 0, 010\rangle$	$J_1 = \langle\langle 0, 000\rangle, \langle 1, 000\rangle, \langle 2, 010\rangle, \langle 0, 010\rangle\rangle$
	$J_2 = \langle\langle 0, 000\rangle, \langle 1, 100\rangle, \langle 0, 100\rangle, \langle 2, 100\rangle, \langle 1, 110\rangle, \langle 0, 010\rangle\rangle$
	$J_3 = \langle\langle 0, 000\rangle, \langle 2, 000\rangle, \langle 1, 010\rangle, \langle 0, 010\rangle\rangle$
	$J_4 = \langle\langle 0, 000\rangle, \langle 2, 001\rangle, \langle 1, 001\rangle, \langle 0, 001\rangle, \langle 1, 101\rangle, \langle 0, 101\rangle, \langle 2, 101\rangle, \langle 1, 111\rangle, \langle 2, 111\rangle, \langle 0, 110\rangle, \langle 2, 110\rangle, \langle 0, 111\rangle, \langle 1, 011\rangle, \langle 0, 011\rangle, \langle 2, 011\rangle, \langle 0, 010\rangle\rangle$
$\langle 0, 110\rangle$	$J_1 = \langle\langle 0, 000\rangle, \langle 1, 000\rangle, \langle 2, 010\rangle, \langle 0, 010\rangle, \langle 2, 011\rangle, \langle 0, 011\rangle, \langle 1, 011\rangle, \langle 0, 111\rangle, \langle 2, 110\rangle, \langle 0, 110\rangle\rangle$
	$J_2 = \langle\langle 0, 000\rangle, \langle 1, 100\rangle, \langle 0, 100\rangle, \langle 2, 100\rangle, \langle 1, 110\rangle, \langle 0, 110\rangle\rangle$
	$J_3 = \langle\langle 0, 000\rangle, \langle 2, 000\rangle, \langle 1, 010\rangle, \langle 0, 110\rangle\rangle$
	$J_4 = \langle\langle 0, 000\rangle, \langle 2, 001\rangle, \langle 1, 001\rangle, \langle 0, 001\rangle, \langle 1, 101\rangle, \langle 0, 101\rangle, \langle 2, 101\rangle, \langle 1, 111\rangle, \langle 2, 111\rangle, \langle 0, 110\rangle\rangle$
$\langle 0, 001\rangle$	$J_1 = \langle\langle 0, 000\rangle, \langle 1, 000\rangle, \langle 2, 010\rangle, \langle 1, 010\rangle, \langle 0, 010\rangle, \langle 2, 011\rangle, \langle 1, 001\rangle, \langle 0, 001\rangle\rangle$
	$J_2 = \langle\langle 0, 000\rangle, \langle 1, 100\rangle, \langle 0, 100\rangle, \langle 2, 100\rangle, \langle 0, 101\rangle, \langle 2, 101\rangle, \langle 1, 111\rangle, \langle 0, 011\rangle, \langle 1, 011\rangle, \langle 0, 111\rangle, \langle 2, 110\rangle, \langle 1, 110\rangle, \langle 0, 110\rangle, \langle 2, 111\rangle, \langle 1, 101\rangle, \langle 0, 001\rangle\rangle$
	$J_3 = \langle\langle 0, 000\rangle, \langle 2, 000\rangle, \langle 0, 001\rangle\rangle$
	$J_4 = \langle\langle 0, 000\rangle, \langle 2, 001\rangle, \langle 0, 001\rangle\rangle$
$\langle 0, 101\rangle$	$J_1 = \langle\langle 0, 000\rangle, \langle 1, 000\rangle, \langle 2, 010\rangle, \langle 0, 011\rangle, \langle 2, 011\rangle, \langle 1, 011\rangle, \langle 0, 111\rangle, \langle 1, 111\rangle, \langle 2, 101\rangle, \langle 0, 101\rangle\rangle$
	$J_2 = \langle\langle 0, 000\rangle, \langle 1, 100\rangle, \langle 0, 100\rangle, \langle 2, 100\rangle, \langle 0, 101\rangle\rangle$
	$J_3 = \langle\langle 0, 000\rangle, \langle 2, 000\rangle, \langle 1, 010\rangle, \langle 0, 010\rangle, \langle 1, 110\rangle, \langle 2, 110\rangle, \langle 0, 110\rangle, \langle 2, 111\rangle, \langle 1, 101\rangle, \langle 0, 101\rangle\rangle$
	$J_4 = \langle\langle 0, 000\rangle, \langle 2, 001\rangle, \langle 0, 001\rangle, \langle 1, 001\rangle, \langle 0, 101\rangle\rangle$
$\langle 0, 011\rangle$	$J_1 = \langle\langle 0, 000\rangle, \langle 1, 000\rangle, \langle 2, 010\rangle, \langle 0, 011\rangle\rangle$
	$J_2 = \langle\langle 0, 000\rangle, \langle 1, 100\rangle, \langle 2, 100\rangle, \langle 0, 100\rangle, \langle 2, 101\rangle, \langle 0, 101\rangle, \langle 1, 101\rangle, \langle 2, 111\rangle, \langle 1, 111\rangle, \langle 0, 011\rangle\rangle$
	$J_3 = \langle\langle 0, 000\rangle, \langle 2, 000\rangle, \langle 1, 010\rangle, \langle 0, 010\rangle, \langle 1, 110\rangle, \langle 0, 110\rangle, \langle 2, 110\rangle, \langle 0, 111\rangle, \langle 1, 111\rangle, \langle 0, 011\rangle\rangle$
	$J_4 = \langle\langle 0, 000\rangle, \langle 2, 001\rangle, \langle 0, 001\rangle, \langle 1, 001\rangle, \langle 2, 011\rangle, \langle 0, 011\rangle\rangle$
$\langle 0, 111\rangle$	$J_1 = \langle\langle 0, 000\rangle, \langle 1, 000\rangle, \langle 2, 010\rangle, \langle 1, 010\rangle, \langle 0, 110\rangle, \langle 1, 110\rangle, \langle 0, 010\rangle, \langle 2, 011\rangle, \langle 0, 011\rangle, \langle 1, 111\rangle, \langle 0, 111\rangle\rangle$
	$J_2 = \langle\langle 0, 000\rangle, \langle 1, 100\rangle, \langle 2, 110\rangle, \langle 0, 111\rangle\rangle$
	$J_3 = \langle\langle 0, 000\rangle, \langle 2, 000\rangle, \langle 0, 001\rangle, \langle 1, 001\rangle, \langle 0, 101\rangle, \langle 2, 100\rangle, \langle 0, 100\rangle, \langle 2, 101\rangle, \langle 1, 101\rangle, \langle 2, 111\rangle, \langle 0, 111\rangle\rangle$
	$J_4 = \langle\langle 0, 000\rangle, \langle 2, 001\rangle, \langle 1, 011\rangle, \langle 0, 111\rangle\rangle$



(a) Subcase 1.1
(b) Subcase 1.2
(c) Subcase 1.3
(d) Subcase 1.4

Figure 4: Illustrations for Case 1 of Lemma 13.

Subcase 1.2: Assume that $ij = 10$. By Lemma 9, there is a 3^*-container $\{P_1, P_2, P_3\}$ of $BF(n-2)$ joining $s' = \langle 0, 0^{n-2}\rangle$ to $d' = \langle 0, x\rangle$ such that only P_1 begins with a level-$(n-3)$ edge. By Corollary 3, there is another 3^*-container $\{Q_1, Q_2, Q_3\}$ of $BF(n-2)$ joining s' to d' such that only Q_1 ends up with a level-$(n-3)$ edge. Moreover, $(I(P_2) \cup I(P_3)) \cap (I(Q_2) \cup I(Q_3))$ contains a level-0 vertex u incident with a level-0 edge and a level-$(n-3)$ edge. We set $u = \langle 0, y\rangle$ with some $y \in \mathbb{Z}_2^{n-2} - \{0^{n-2}, x\}$. Further, we set $u_h^{pq} = \langle h, pqy\rangle$ and $d_h^{pq} = \langle h, pqx\rangle$ for any $p, q \in \mathbb{Z}_2$ and $h \in \{0, 1, 2\}$. Let Γ_1 be the subgraph of $BF(n-2)$ generated by $E(Q_1) \cup E(Q_2) \cup E(Q_3)$. By Corollary 3, Γ_1 is 0-designed with respect to $BF(n-2)$. Thus, $\gamma_1^0 \circ \gamma_0^0(\Gamma_1)$ spans $BF_{0,1}^{0,0}(n)$ and consists of three disjoint paths $\{D_1^{00}, D_2^{00}, D_3^{00}\}$ joining s to d_2^{00}. Accordingly, we have $D_1^{00} = \langle s, T_{00}, d_0^{00}, d_1^{00}, d_2^{00}\rangle$ where T_{00} joins s to d_0^{00}. Moreover, D_2^{00} and D_3^{00} form a cycle which can be written as $\langle s, W_{00}, u_0^{00}, u_1^{00}, u_2^{00}, U_{00}, d_2^{00}, A_{00}, s\rangle$.

Suppose that Γ_2 is the subgraph of $BF(n-2)$ generated by $E(P_1) \cup E(P_2) \cup E(P_3)$. By Lemma 9, Γ_2 is 0-designed with respect to $BF(n-2)$. Hence, $\gamma_1^j \circ \gamma_0^i(\Gamma_2)$ spans $BF_{0,1}^{i,j}(n)$ and consists of three disjoint paths $\{D_1^{ij}, D_2^{ij}, D_3^{ij}\}$ between s_2^{ij} and d. Accordingly, we write $D_1^{ij} = \langle s_2^{ij}, s_1^{ij}, s_0^{ij}, T_{ij}, d\rangle$ where T_{ij} joins s_0^{ij} to d. Moreover, D_2^{ij} and D_3^{ij} form a cycle which can be written as $\langle\, s_2^{ij}, W_{ij}, u_0^{ij}, u_1^{ij}, u_2^{ij}, U_{ij}, d, A_{ij}, s_2^{ij}\,\rangle$. By Lemma 4, there is a totally scheduled hamiltonian cycle $C^{pq} = \langle\, s_0^{pq}, s_1^{pq}, s_2^{pq}, R_{pq}, u_0^{pq}, u_1^{pq}, u_2^{pq}, D_{pq}, d_0^{pq}, d_1^{pq}, d_2^{pq}, H_{pq}, s_0^{pq}\,\rangle$ of $BF_{0,1}^{p,q}(n)$ for $pq \in \mathbb{Z}_2^2 - \{00, ij\}$. Then we create a 4^*-container $\{J_1, J_2, J_3, J_4\}$ of $BF(n)$, in which $J_1 = \langle s, T_{00}, d_0^{00}, d_1^{00}, d\rangle$, $J_2 = \langle s, A_{00}^{-1}, d_2^{00}, U_{00}^{-1}, u_2^{00}, u_1^{00}, u_0^{10}, W_{10}^{-1}, s_2^{10}, A_{10}^{-1}, d\rangle$, $J_3 = \langle s, W_{00}, u_0^{00}, u_1^{10}, u_2^{11}, D_{11}, d_0^{11}, d_1^{11}, d_2^{11}, H_{11}, s_0^{11}, s_1^{11}, s_2^{11}, R_{11}, u_0^{11}, u_1^{01}, u_2^{01}, D_{01}, d_0^{01}, d_1^{01}, d_2^{01}, H_{01}, s_0^{01}, s_1^{01}, s_2^{01}, R_{01}, u_0^{01}, u_1^{11}, u_2^{10}, U_{10}, d\rangle$, and $J_4 = \langle s, s_1^{10}, s_0^{10}, T_{10}, d\rangle$. See Figure 4(b).

Subcase 1.3: Assume that $ij = 01$. Similar to **Subcase 1.2**, we create a 4^*-container $\{J_1, J_2, J_3, J_4\}$ of $BF(n)$, in which $J_1 = \langle s, T_{00}, d_0^{00}, d_1^{00}, d_0^{10}, D_{10}^{-1}, u_2^{10}, u_1^{10}, u_0^{10}, R_{10}^{-1}, s_2^{10}, s_1^{11}, s_2^{11}, R_{11}, u_0^{11}, u_1^{11}, u_2^{11}, D_{11}, d_0^{11}, d_1^{11}, d\rangle$, $J_2 = \langle s, A_{00}^{-1}, d_2^{00}, U_{00}^{-1}, u_2^{00}, u_1^{01}, u_0^{01}, W_{01}^{-1}, s_2^{01}, A_{01}^{-1}, d\rangle$, $J_3 = \langle s, W_{00}, u_0^{00}, u_1^{00}, u_2^{01}, U_{01}, d\rangle$, and $J_4 = \langle s, s_1^{10}, s_0^{10}, H_{10}^{-1}, d_2^{10}, d_1^{10}, d_2^{11}, H_{11}, s_0^{11}, s_1^{01}, s_0^{01}, T_{01}, d\rangle$. See Figure 4(c).

Subcase 1.4: Assume that $ij = 11$. Similar to **Subcase 1.2**, we create a 4^*-container $\{J_1, J_2, J_3, J_4\}$ of $BF(n)$, in which $J_1 = \langle s, T_{00}, d_0^{00}, d_1^{00}, d_2^{01}, H_{01}, s_0^{01}, s_1^{01}, s_2^{01}, R_{01}, u_0^{01}, u_1^{01}, u_0^{11}, W_{11}^{-1}, s_2^{11}, A_{11}^{-1}, d\rangle$, $J_2 = \langle s, A_{00}^{-1}, d_2^{00}, U_{00}^{-1}, u_2^{00}, u_1^{00}, u_2^{01}, D_{01}, d_0^{01}, d_1^{01}, d\rangle$, $J_3 = \langle s, W_{00}, u_0^{00}, u_1^{10}, u_0^{10}, R_{10}^{-1}, s_2^{10}, s_1^{11}, s_0^{11}, T_{11}, d\rangle$, and $J_4 = \langle s, s_1^{10}, s_0^{10}, H_{10}^{-1}, d_2^{10}, d_1^{10}, d_0^{10}, D_{10}^{-1}, u_2^{10}, u_1^{11}, u_2^{11}, U_{11}, d\rangle$. See Figure 4(d).

Case 2: Suppose that $x = 0^{n-2}$. Let $t_h^{pq} = \langle h,\ pq0^{n-3}1\rangle$, $u_h^{pq} = \langle h,\ pq0^{n-2}\rangle$, and $w^{pq} = \langle n-1,\ pq0^{n-3}1\rangle$ for any $p, q \in \mathbb{Z}_2$ and $h \in \{0, 1, 2\}$. Obviously, $s = u_0^{00}$, $d = u_0^{ij}$, and $\{(s, w^{00}), (d, w^{ij})\} \subset E(BF(n))$. Suppose that $C_0^{00} = \langle s, u_1^{00}, u_2^{00}, D_{00}, s\rangle$ is a cycle of length n, in which $D_{00} = \langle u_2^{00}, g(u_2^{00}), \ldots, g^{n-3}(u_2^{00})\rangle$. By Lemma 6, there is a totally scheduled hamiltonian cycle $C_1^{00} = \langle w^{00}, t_0^{00}, t_1^{00}, t_2^{00}, R_{00}, w^{00}\rangle$ of $BF_{0,1}^{0,0}(n) - V(C_0^{00})$. By Lemma 7, there is a hamiltonian cycle $\langle d, u_1^{ij}, u_2^{ij}, R_{ij}, d\rangle$ of $BF_{0,1}^{i,j}(n) - \{w^{ij}, u_0^{ij}, u_1^{ij}\}$ and for $pq \in \mathbb{Z}_2^2 - \{00, ij\}$, there is a hamiltonian cycle $O^{pq} = \langle t_0^{pq}, t_1^{pq}, t_2^{pq}, T_{pq}, t_0^{pq}\rangle$ of $BF_{0,1}^{p,q}(n) - \{u_1^{pq}\}$. By Lemma 4, there is a totally scheduled hamiltonian cycle $\langle u_0^{pq}, u_1^{pq}, u_2^{pq}, A_{pq}, u_0^{pq}\rangle$ of $BF_{0,1}^{p,q}(n)$ for any p, q.

Subcase 2.1: Assume that $ij = 10$. We create a 4^*-container $\{J_1, J_2, J_3, J_4\}$ of $BF(n)$, in which $J_1 = \langle s,\ u_1^{10},\ d\rangle$, $J_2 = \langle s, u_1^{00}, d\rangle$, $J_3 = \langle s,\ D_{00}^{-1},\ u_2^{00},\ u_1^{01},\ u_0^{11},\ A_{11},\ u_2^{11},\ u_1^{11},\ u_2^{10},\ R_{10},\ d\rangle$, and $J_4 = \langle s,\ w^{00},\ R_{00}^{-1},\ t_2^{00},\ t_1^{01},\ t_0^{01},\ T_{01}^{-1},\ t_2^{01},\ t_1^{00},\ t_0^{00},\ t_1^{10},\ t_0^{10},\ w^{10},\ d\rangle$. See Figure 5(a).

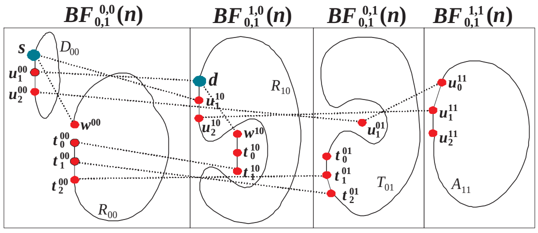
Subcase 2.2: Assume that $ij = 01$. Let $Q_0^{11} = \langle u_0^{11}, u_1^{11}, u_2^{11}, D_{11}, u_0^{11}\rangle$ be a cycle of length n, in which $D_{11} = \langle u_2^{11}, g(u_2^{11}), \ldots, g^{n-3}(u_2^{11})\rangle$. By Lemma 6, there is a totally scheduled hamiltonian cycle $Q_1^{11} = \langle t_0^{11}, t_1^{11}, t_2^{11}, U_{11}, t_0^{11}\rangle$ of $BF_{0,1}^{1,1}(n) - V(Q_0^{11})$. Then we build a 4^*-container $\{J_1, J_2, J_3, J_4\}$ of $BF(n)$, in which $J_1 = \langle s, u_1^{10}, u_2^{11}, D_{11}, u_0^{11}, u_1^{11}, d\rangle$, $J_2 = \langle s, u_1^{00}, u_2^{01}, R_{01}, d\rangle$, $J_3 = \langle s, D_{00}^{-1}, u_2^{00}, u_1^{01}, d\rangle$, and $J_4 = \langle s,\ w^{00},\ R_{00}^{-1},\ t_2^{00},\ t_1^{00},\ t_0^{00},\ t_1^{10},\ t_0^{10},\ T_{10}^{-1},\ t_2^{10},\ t_1^{11},\ t_2^{11},\ U_{11},\ t_0^{11},\ t_1^{01},\ t_0^{01},\ w^{01},\ d\rangle$. See Figure 5(b).

Subcase 2.3: Assume that $ij = 11$. We create a 4^*-container $\{J_1, J_2, J_3, J_4\}$ of $BF(n)$, in which $J_1 = \langle s, u_1^{10}, u_2^{11}, R_{11}, d\rangle$, $J_2 = \langle s, u_1^{00}, u_2^{01}, A_{01}, u_0^{01}, u_1^{11}, d\rangle$, $J_3 = \langle s,\ D_{00}^{-1},\ u_2^{00},\ u_1^{01},\ d\rangle$, and $J_4 = \langle s,\ w^{00},\ R_{00}^{-1},\ t_2^{00},\ t_1^{00},\ t_0^{00},\ t_1^{10},\ t_0^{10},\ T_{10}^{-1},\ t_2^{10},\ t_1^{11},\ t_0^{11},\ w^{11},\ d\rangle$. See Figure 5(c). □

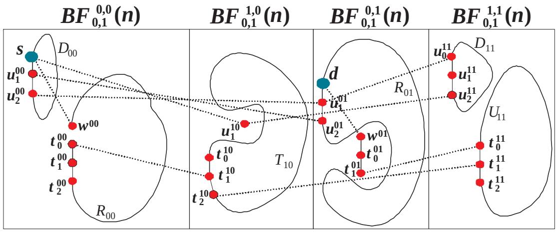
Lemma 14. *For $n \geq 3$, assume that ℓ_s and ℓ_d are integers of $\mathbb{Z}_n$ such that $1 \leq \ell_d - \ell_s \leq n-1$. Let s be any level-ℓ_s vertex of $BF(n)$ and d be any level-ℓ_d vertex of $BF(n)$. Then there is a 4^*-container Ω of $BF(n)$ joining s to d such that the following requirements are satisfied: (i) at least one path of Ω, ending with a level-$(\ell_d - 1)_{\bmod n}$ edge, is weakly ℓ_d-scheduled, and (ii) at least one path of Ω, ending up with a level-ℓ_d edge, is weakly ℓ_d-scheduled.*

Proof. Without loss of generality, we assume $\ell_s = 0$ such that $s = \langle 0, 0^n\rangle$ and $d = \langle \ell_d, x_1 ij x_2\rangle$ with some $1 \leq \ell_d \leq n-1$, $i, j \in \mathbb{Z}_2$, $x_1 \in \mathbb{Z}_2^{\ell_d}$, and $x_2 \in \mathbb{Z}_2^{n-2-\ell_d}$. We prove this lemma by induction on n. Note that we only construct the desired container for $1 \leq \ell_d \leq \lfloor n/2 \rfloor$. By symmetry of $BF(n)$, the required container for $\lfloor n/2 \rfloor + 1 \leq \ell_d \leq n-1$ can be easily derived. Moreover, only odd ℓ_d will be concerned when n is even. The induction bases, depending on $n \in \{3, 4\}$, can be checked by a computer program. As the induction hypothesis, we suppose the statement holds for $BF(n-2)$ with $n \geq 5$. Then we partition $BF(n)$ into $\{BF_{\ell_d, \ell_d+1}^{p,q}(n) \mid p, q \in \mathbb{Z}_2\}$.

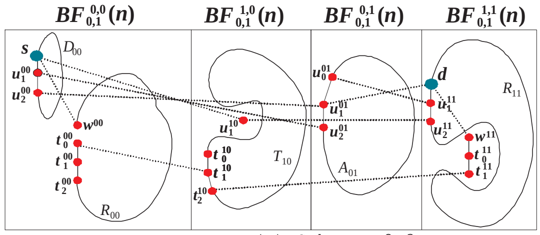
By the induction hypothesis, there is a 4^*-container $\{P_1', P_2', P_3', P_4'\}$ of $BF(n-2)$ joining $s' = \langle 0, 0^{n-2}\rangle$ to $d' = \langle \ell_d, x_1 x_2\rangle$ such that (1) P_1' is weakly ℓ_d-scheduled and ends up with a level-$(\ell_d - 1)_{\bmod n}$ edge,



(a) Subcase 2.1



(b) Subcase 2.2



(c) Subcase 2.3

Figure 5: Illustrations for Case 2 of Lemma 13.

(2) P'_3 and P'_4 ends up with level-ℓ_d edges, and (3) P'_4 is weakly ℓ_d-scheduled. That is, at least one level-ℓ_d vertex of $I(P'_1)$, $u' = \langle \ell_d, y_1y_2\rangle$ with $|y_1| = |x_1|$ and $|y_2| = |x_2|$, is incident with only one level-ℓ_d edge on P'_1. Similarly, there is at least one level-ℓ_d vertex of $I(P'_4)$, $t' = \langle \ell_d, z_1z_2\rangle$ with $|z_1| = |x_1|$ and $|z_2| = |x_2|$, incident with only one level-ℓ_d edge on P'_4. Let Γ be the subgraph of $BF(n-2)$ generated by $E(P'_1) \cup E(P'_2) \cup E(P'_3) \cup E(P'_4)$. Let $d_h^{pq} = \langle \ell_d + h, x_1ijx_2\rangle$, $u_h^{pq} = \langle \ell_d + h, y_1pqy_2\rangle$, and $t_h^{pq} = \langle \ell_d + h, z_1pqz_2\rangle$ for any $p, q \in \mathbb{Z}_2$, $h \in \{-1, 0, 1, 2\}$. Obviously, $\{(u_0^{00}, u_1^{00}), (u_1^{00}, u_2^{00})\} \subset E(\gamma^0_{\ell_d+1} \circ \gamma^0_{\ell_d}(P'_1))$ and $\{(t_0^{00}, t_1^{00}), (t_1^{00}, t_2^{00})\} \subset E(\gamma^0_{\ell_d+1} \circ \gamma^0_{\ell_d}(P'_4))$. Moreover, $\gamma^0_{\ell_d+1} \circ \gamma^0_{\ell_d}(P'_3)$ and $\gamma^0_{\ell_d+1} \circ \gamma^0_{\ell_d}(P'_4)$ form a cycle which can be written as $\langle s, Q_{00}, t_0^{00}, t_1^{00}, t_2^{00}, H_{00}, d_2^{00}, A_{00}, s\rangle$.

Case 1: Suppose that $ij = 00$. By Corollary 2, there is a hamiltonian path $\langle t_0^{10}, T_{10}, d_1^{10}\rangle$ of $BF^{1,0}_{\ell_d,\ell_d+1}(n)$. By Lemma 5, there is a hamiltonian path $\langle t_1^{01}, t_0^{01}, W_{01}, d_2^{01}\rangle$ of $BF^{0,1}_{\ell_d,\ell_d+1}(n)$. By Lemma 4, there is a totally scheduled hamiltonian cycle $C^{11} = \langle t_0^{11}, t_1^{11}, t_2^{11}, R_{11}, t_0^{11}\rangle$ of $BF^{1,1}_{\ell_d,\ell_d+1}(n)$. Then $P_1 = \gamma^0_{\ell_d+1} \circ \gamma^0_{\ell_d}(P'_1)$, $P_2 = \gamma^0_{\ell_d+1} \circ \gamma^0_{\ell_d}(P'_2)$, $P_3 = \langle s, A_{00}^{-1}, d_2^{00}, H_{00}^{-1}, t_2^{00}, t_1^{01}, t_0^{11}, R_{11}^{-1}, t_2^{11}, t_1^{11}, t_0^{01}, W_{01}, d_2^{01}, d_1^{00}, d\rangle$, and $P_4 = \langle s, Q_{00}, t_0^{00}, t_1^{00}, t_0^{10}, T_{10}, d_1^{10}, d\rangle$ are four disjoint paths joining s to d. See Figure 6(a) for illustration.

Case 2: Suppose that $ij \neq 00$. Let $\tau = \langle \ell_d - 1, z_1z_2\rangle$. By Lemma 10, there is a 3^*-container $\{S'_1, S'_2, S'_3\}$ of $BF(n-2)$ joining τ to d' such that $(\tau, g(\tau)) \in E(S'_3)$ and also that S'_1 ends up with a level-ℓ_d edge. Let Θ be the subgraph of $BF(n-2)$ generated by $E(S'_1) \cup E(S'_2) \cup E(S'_3)$. According to Lemma 10, Θ is ℓ_d-designed. Therefore, $\gamma^j_{\ell_d+1} \circ \gamma^i_{\ell_d}(\Theta)$ spans $BF^{i,j}_{\ell_d,\ell_d+1}(n)$. Then we have $\gamma^j_{\ell_d+1} \circ \gamma^i_{\ell_d}(S'_1) = \langle t_{-1}^{ij}, R_{ij}, d_2^{ij}, d_1^{ij}, d\rangle$, $\gamma^j_{\ell_d+1} \circ \gamma^i_{\ell_d}(S'_2) = \langle t_{-1}^{ij}, D_{ij}, d\rangle$, and $\gamma^j_{\ell_d+1} \circ \gamma^i_{\ell_d}(S'_3) = \langle t_{-1}^{ij}, t_0^{ij}, t_1^{ij}, t_2^{ij}, H_{ij}, d\rangle$ for some paths R_{ij}, D_{ij}, and H_{ij}. For $pq \in \mathbb{Z}_2^2 - \{00, ij\}$, there is a totally scheduled hamiltonian cycle C^{pq} of $BF^{p,q}_{\ell_d,\ell_d+1}(n)$ by Lemma 4 and there is a hamiltonian path $\langle t_1^{pq}, T_{pq}, d_2^{pq}\rangle$ of $BF^{p,q}_{\ell_d,\ell_d+1}(n)$ by Lemma 5.

Subcase 2.1: If $ij = 10$, $d = d_0^{10}$. By Corollary 2, there is a hamiltonian path S_{11} of $BF^{1,1}_{\ell_d,\ell_d+1}(n)$ joining t_1^{11} to d_0^{11} such that $(u_0^{11}, u_1^{11}) \in E(S_{11})$. We set $A = \{ (u_1^{00}, u_2^{01}), (u_2^{00}, u_1^{01}), (t_0^{00}, t_1^{10}), (t_1^{00}, t_0^{10}), (t_1^{00}, t_2^{01}), (t_2^{00}, t_1^{01}), (d_1^{00}, d_2^{01}), (d_1^{00}, d_0^{10}), (d_2^{10}, d_1^{11}), (d_1^{10}, d_2^{11}), (t_2^{10}, t_1^{11}), (u_0^{01}, u_1^{11}), (u_1^{01}, u_0^{11}), (d_1^{01}, d_0^{11})\}$ and $B = \{(u_1^{00}, u_2^{00}), (t_0^{00}, t_1^{00}), (t_1^{00}, t_2^{00}), (d_0^{00}, d_1^{00}), (d_1^{00}, d_2^{00}), (d_1^{10}, d_2^{10}), (t_{-1}^{10}, t_0^{10}), (t_1^{10}, t_2^{10}), (t_1^{01}, t_2^{01}), (u_0^{01}, u_1^{01}), (u_1^{01}, u_2^{01}), (d_1^{01}, d_2^{01}), (u_0^{11}, u_1^{11}), (d_1^{11}, d_2^{11})\}$. Then a 4-container $\{P_1, P_2, P_3, P_4\}$ of $BF(n)$ between s and d can be formed from the subgraph generated by $(E(\gamma^0_{\ell_d+1} \circ \gamma^0_{\ell_d}(\Gamma)) \cup E(\gamma^0_{\ell_d+1} \circ \gamma^1_{\ell_d}(\Theta)) \cup E(C^{01}) \cup E(S_{11}) \cup A) - B$. See Figure 6(b) for illustration.

Subcase 2.2: If $ij = 01$, $d = d_0^{01}$. We set $A = \{(u_0^{00}, u_1^{10}), (u_1^{00}, u_0^{10}), (t_1^{00}, t_2^{01}), (t_2^{00}, t_1^{01}), (d_1^{00}, d_2^{01}), (d_1^{00}, d_0^{10}), (d_1^{10}, d_2^{11}), (d_1^{01}, d_0^{11}), (d_0^{01}, d_1^{11}), (t_0^{01}, t_1^{11})\}$ and set $B = \{(u_0^{00}, u_1^{00}), (t_1^{00}, t_2^{00}), (d_0^{00}, d_1^{00}), (d_1^{00}, d_2^{00}), (u_0^{10}, u_1^{10}), (d_0^{10}, d_1^{10}), (t_{-1}^{01}, t_0^{01}), (t_1^{01}, t_2^{01}), (d_1^{01}, d_2^{01}), (d_0^{11}, d_1^{11})\}$. A 4-container $\{P_1, P_2, P_3, P_4\}$ of $BF(n)$ between s and d can be formed from the subgraph generated by $(E(\gamma^0_{\ell_d+1} \circ \gamma^0_{\ell_d}(\Gamma)) \cup E(\gamma^1_{\ell_d+1} \circ \gamma^0_{\ell_d}(\Theta)) \cup E(C^{10}) \cup E(T_{11}) \cup A) - B$. See Figure 6(c).

Subcase 2.3: If $ij = 11$, $d = d_0^{11}$. We set $A = \{(u_0^{00}, u_1^{10}), (u_1^{00}, u_0^{10}), (t_1^{00}, t_0^{10}), (t_2^{00}, t_1^{01}), (d_1^{00}, d_2^{01}), (d_1^{00}, d_0^{10}), (d_1^{10}, d_2^{11}), (t_1^{10}, t_2^{11}), (t_0^{01}, t_1^{11}), (t_1^{01}, t_0^{11}), (d_0^{01}, d_1^{11}), (d_1^{01}, d_0^{11})\}$, and set $B = \{(u_0^{00}, u_1^{00}), (t_1^{00}, t_2^{00}), (d_0^{00}, d_1^{00}), (d_1^{00}, d_2^{00}), (u_0^{10}, u_1^{10}), (d_0^{10}, d_1^{10}), (t_0^{10}, t_1^{10}), (t_{-1}^{11}, t_0^{11}), (t_1^{11}, t_2^{11}), (d_1^{11}, d_2^{11}), (d_0^{01}, d_1^{01}), (t_0^{01}, t_1^{01})\}$. Then a 4-container $\{P_1, P_2, P_3, P_4\}$ of $BF(n)$ between s and d can be formed from the subgraph generated by $(E(\gamma^0_{\ell_d+1} \circ \gamma^0_{\ell_d}(\Gamma)) \cup E(\gamma^1_{\ell_d+1} \circ \gamma^1_{\ell_d}(\Theta)) \cup E(C^{10}) \cup E(T_{01}) \cup A) - B$. See Figure 6(d) for illustration.

By Lemma 2, $V(BF(n)) - \bigcup_{i=1}^{4} V(P_i) = V(BF^{0,0}_{\ell_d,\ell_d+1}) - V(\gamma^0_{\ell_d+1} \circ \gamma^0_{\ell_d}(\Gamma)) = \bigcup_{i=1}^{m}\{u_i, g(u_i)\}$ for some $m \geq 1$. Moreover, $\bigcup_{i=1}^{m}\{(f(u_i), g^{-1} \circ$

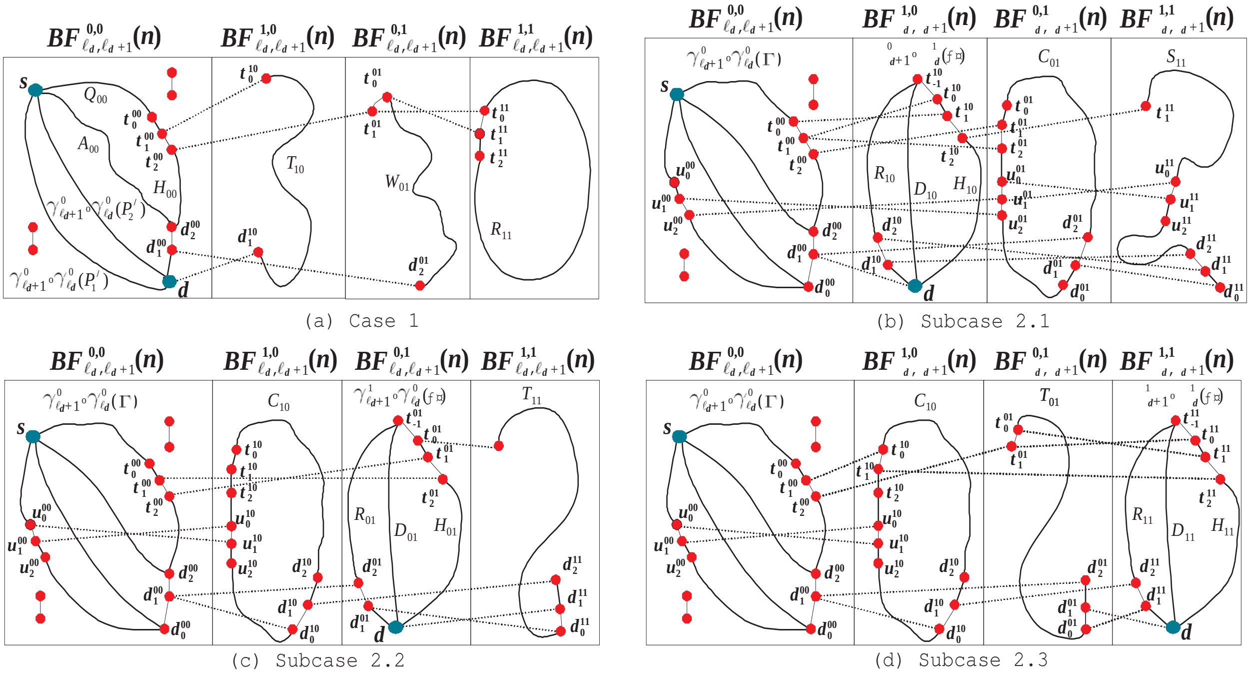


Figure 6: Illustrations for Lemma 14.

$f(u_i))\} \subseteq \bigcup_{i=1}^{4} E(P_i)$. Thus, by Lemma 3, there is a 4^*-container of $BF(n)$ between s and d. $\square$

According to Lemma 13 and Lemma 14, we derive the following theorem.

Theorem 4. *Let $n \geq 3$. Then $BF(n)$ is 4^*-connected if n is odd and is 4^*-laceable otherwise.*

Combining Theorem 1, Theorem 2, Theorem 3, and Theorem 4, we summarize the main result as follows.

Theorem 5. *Let $n \geq 3$. Then $BF(n)$ is super spanning connected if n is odd, and is super spanning laceable otherwise.*

6 Conclusion

This paper is aimed to show that $BF(n)$ is super spanning connected for n odd and super spanning laceable for n even. A k-container $C_k(u, v)$ between two distinct vertices u and v in G is a set of k disjoint paths between u and v. The length of a $C_k(u, v)$, written as $l(C_k(u, v))$, is the length of the longest path in $C_k(u, v)$. The k-wide distance between u and v is $d_k(u, v)$, which is the minimum length among all k-containers between u and v. Let κ be the connectivity of a graph G. The wide diameter of G, denoted by $D_\kappa(G)$, is the maximum of κ-wide distances among all pairs of distinct vertices in G. Assume that G is k^*-connected. We may define the k^*-wide distance between any two vertices u and v, denoted by $d_k^*(u, v)$, to be the minimum length among all k^*-containers between u and v. Let $D_k^*(G) = \max\{d_k^*(u, v) \mid u$ and v are two different vertices of $G\}$. We say that $D_k^*(G)$ is the k^*-diameter of G. In our future work, we are interested in computing $D_k^*(BF(n))$ for $n \geq 3$.

Appendix

A Proof of Lemma 9

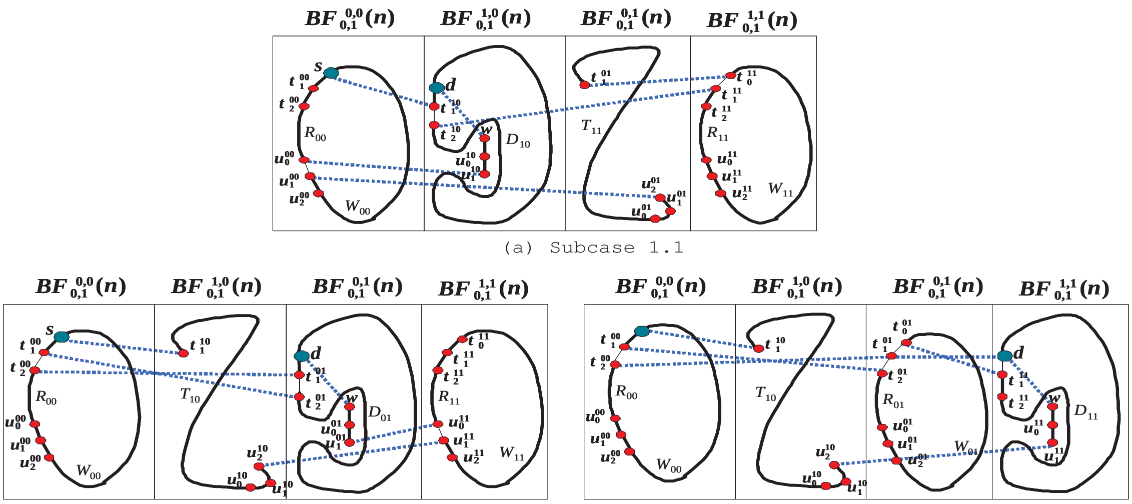
Proof. Without loss of generality, we assume that $\ell = 0$ so that $s = \langle 0, 0^n \rangle$ and $d = \langle 0, ijx \rangle$ for some $i, j \in \mathbb{Z}_2$, and some $x \in \mathbb{Z}_2^{n-2}$. Then this lemma will be proved by induction on n. The induction bases depend upon 3^*-containers of $BF(3)$, listed in Table 4. Next, suppose that the statement holds for $BF(n-2)$ with $n \geq 5$. To create the desired container, we may partition $BF(n)$ into $\{BF_{0,1}^{p,q}(n) \mid p, q \in \mathbb{Z}_2\}$.

Case 1: Suppose that $x = 0^{n-2}$. Let $w = f^{-1}(d) = \langle n-1, ij0^{n-3}1 \rangle$. Moreover, let $t_h^{pq} = \langle h, pq0^{n-2} \rangle$ and $u_h^{pq} = \langle h, pq0^{n-3}1 \rangle$ for any $p, q \in \mathbb{Z}_2$ and $h \in \{0, 1, 2\}$. Thus, $s = t_0^{00}$. By Lemma 7, there is a hamiltonian cycle $Q^{ij} = \langle d, t_1^{ij}, t_2^{ij}, D_{ij}, d \rangle$ of $BF_{0,1}^{i,j}(n) - \{w, u_0^{ij}, u_1^{ij}\}$. For $pq \in \mathbb{Z}_2^2 - \{ij\}$, there is a 0-scheduled hamiltonian path $\langle t_1^{pq}, T_{pq}, u_0^{pq}, u_1^{pq}, u_2^{pq} \rangle$ of $BF_{0,1}^{p,q}(n)$ by Corollary 2 and there is a totally scheduled hamiltonian cycle $C^{pq} = \langle t_0^{pq}, t_1^{pq}, t_2^{pq}, R_{pq}, u_0^{pq}, u_1^{pq}, u_2^{pq}, W_{pq}, t_0^{pq} \rangle$ of $BF_{0,1}^{p,q}(n)$ by Lemma 4. Then we create a 3^*-container $\{P_1, P_2, P_3\}$ of $BF(n)$ as follows.

Subcase 1.1: If $ij = 10$, then $P_1 = \langle s, W_{00}^{-1}, u_2^{00}, u_1^{00}, u_2^{01}, u_1^{01}, u_0^{01}, T_{01}^{-1}, t_1^{01}, t_0^{11}, W_{11}^{-1}, u_2^{11}, u_1^{11}, u_0^{11}, R_{11}^{-1}, t_2^{11}, t_1^{11}, t_2^{10}, D_{10}, d \rangle$, $P_2 = \langle s, t_1^{10}, d \rangle$, and $P_3 = \langle s, t_1^{00}, t_2^{00}, R_{00}, u_0^{00}, u_1^{10}, u_0^{10}, w, d \rangle$. Obviously,

Table 4: 3^*-containers of $BF(3)$ as induction bases. The vertices fitting the fourth requirement are marked by star symbols.

d	3^*-container $\{P_1, P_2, P_3\}$ joining $\langle 0,000\rangle$ to d
$\langle 0,100\rangle$	$P_1 = \langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 2,001\rangle, \langle 1,011\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,111\rangle, \langle 0,111\rangle, \langle 2,111\rangle, \langle 1,101\rangle, \langle 0,101\rangle, \langle 2,101\rangle, \langle 0,100\rangle\rangle$ $P_2 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 0,100\rangle\rangle$ $P_3 = \langle\langle 0,000\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 0,010\rangle^*, \langle 1,010\rangle, \langle 0,110\rangle^*, \langle 2,110\rangle, \langle 1,110\rangle, \langle 2,100\rangle, \langle 0,100\rangle\rangle$
$\langle 0,010\rangle$	$P_1 = \langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 2,001\rangle, \langle 1,011\rangle, \langle 0,111\rangle, \langle 2,111\rangle, \langle 1,111\rangle, \langle 0,011\rangle, \langle 2,011\rangle, \langle 0,010\rangle\rangle$ $P_2 = \langle\langle 0,000\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 0,010\rangle\rangle$ $P_3 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 0,100\rangle^*, \langle 2,101\rangle, \langle 1,101\rangle, \langle 0,101\rangle, \langle 2,100\rangle, \langle 1,110\rangle, \langle 2,110\rangle, \langle 0,110\rangle^*, \langle 1,010\rangle, \langle 0,010\rangle\rangle$
$\langle 0,110\rangle$	$P_1 = \langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 2,001\rangle, \langle 1,011\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,111\rangle, \langle 0,111\rangle, \langle 2,111\rangle, \langle 0,110\rangle\rangle$ $P_2 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 0,100\rangle^*, \langle 2,101\rangle, \langle 1,101\rangle, \langle 0,101\rangle, \langle 2,100\rangle, \langle 1,110\rangle, \langle 2,110\rangle, \langle 0,110\rangle\rangle$ $P_3 = \langle\langle 0,000\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 0,010\rangle^*, \langle 1,010\rangle, \langle 0,110\rangle\rangle$
$\langle 0,001\rangle$	$P_1 = \langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 0,001\rangle\rangle$ $P_2 = \langle\langle 0,000\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 1,010\rangle, \langle 0,110\rangle, \langle 2,110\rangle, \langle 1,110\rangle, \langle 0,010\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,111\rangle, \langle 2,101\rangle, \langle 1,101\rangle, \langle 2,111\rangle, \langle 0,111\rangle, \langle 1,011\rangle, \langle 2,001\rangle, \langle 0,001\rangle\rangle$ $P_3 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 0,100\rangle^*, \langle 2,100\rangle, \langle 0,101\rangle^*, \langle 1,001\rangle, \langle 0,001\rangle\rangle$
$\langle 0,101\rangle$	$P_1 = \langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 0,101\rangle\rangle$ $P_2 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 0,100\rangle, \langle 2,100\rangle, \langle 0,101\rangle\rangle$ $P_3 = \langle\langle 0,000\rangle, \langle 1,000\rangle, \langle 2,000\rangle, \langle 1,010\rangle, \langle 2,010\rangle, \langle 0,010\rangle, \langle 1,110\rangle, \langle 0,110\rangle^*, \langle 2,110\rangle, \langle 0,111\rangle, \langle 1,011\rangle, \langle 2,011\rangle, \langle 0,011\rangle^*, \langle 1,111\rangle, \langle 2,111\rangle, \langle 1,101\rangle, \langle 2,101\rangle, \langle 0,101\rangle\rangle$
$\langle 0,011\rangle$	$P_1 = \langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 2,011\rangle, \langle 0,011\rangle\rangle$ $P_2 = \langle\langle 0,000\rangle, \langle 1,000\rangle, \langle 2,000\rangle, \langle 1,010\rangle, \langle 0,110\rangle, \langle 2,110\rangle, \langle 1,110\rangle, \langle 0,010\rangle, \langle 2,010\rangle, \langle 0,011\rangle\rangle$ $P_3 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 0,100\rangle^*, \langle 2,100\rangle, \langle 0,101\rangle, \langle 1,101\rangle, \langle 2,101\rangle, \langle 1,111\rangle, \langle 2,111\rangle, \langle 0,111\rangle^*, \langle 1,011\rangle, \langle 0,011\rangle\rangle$
$\langle 0,111\rangle$	$P_1 = \langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 0,001\rangle, \langle 1,101\rangle, \langle 2,111\rangle, \langle 0,111\rangle\rangle$ $P_2 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 0,100\rangle^*, \langle 2,100\rangle, \langle 1,110\rangle, \langle 0,110\rangle, \langle 2,110\rangle, \langle 0,111\rangle\rangle$ $P_3 = \langle\langle 0,000\rangle, \langle 1,000\rangle, \langle 2,000\rangle, \langle 1,010\rangle, \langle 0,010\rangle, \langle 2,010\rangle, \langle 0,011\rangle^*, \langle 1,011\rangle, \langle 2,011\rangle, \langle 1,001\rangle, \langle 0,101\rangle, \langle 2,101\rangle, \langle 1,111\rangle, \langle 0,111\rangle\rangle$



(a) Subcase 1.1

(b) Subcase 1.2

(c) Subcase 1.3

Figure 7: Illustrations for Case 1 of Lemma 9.

$u_0^{00} \in I(P_3)$ and $\mu_{s,d}(u_0^{00}) = u_0^{10} \in I(P_3)$. See Figure 7(a).

Subcase 1.2: If $ij = 01$, then $P_1 = \langle s, W_{00}^{-1}, u_2^{00}, u_1^{00}, u_0^{00}, R_{00}^{-1}, t_2^{00}, t_1^{01}, d\rangle$, $P_2 = \langle s, t_1^{00}, t_2^{01}, D_{01}, d\rangle$, and $P_3 = \langle s, t_1^{10}, T_{10}, u_0^{10}, u_1^{10}, u_2^{10}, u_1^{11}, u_2^{11}, W_{11}, t_0^{11}, t_1^{11}, t_2^{11}, R_{11}, u_0^{11}, u_1^{01}, u_0^{01}, w, d\rangle$. Obviously, $u_0^{10} \in I(P_3)$ and $\mu_{s,d}(u_0^{10}) = u_0^{11} \in I(P_3)$. See Figure 7(b).

Subcase 1.3: If $ij = 11$, then $P_1 = \langle s, W_{00}^{-1}, u_2^{00}, u_1^{00}, u_0^{00}, R_{00}^{-1}, t_2^{00}, t_1^{01}, d\rangle$, $P_2 = \langle s, t_1^{00}, t_2^{01}, R_{01}, u_0^{01}, u_1^{01}, u_2^{01}, W_{01}, t_0^{01}, t_1^{11}, t_2^{11}, D_{11}, d\rangle$, and $P_3 = \langle s, t_1^{10}, T_{10}, u_0^{10}, u_1^{10}, u_2^{10}, u_1^{11}, u_0^{11}, w, d\rangle$. Obviously, $u_0^{10} \in I(P_3)$ and $\mu_{s,d}(u_0^{10}) = u_0^{01} \in I(P_2)$. See Figure 7(c).

Case 2: Suppose that $x \neq 0^{n-2}$. By induction hypothesis, there is a 3*-container $\{P_1', P_2', P_3'\}$ of $BF(n-2)$ joining $s' = \langle 0, 0^{n-2}\rangle$ to $d' = \langle 0, x\rangle$ such that P_1' begins with a level-$(n-3)$ edge and also that there is a level-0 vertex $t = \langle 0, y\rangle \in I(P_3')$ incident with a level-0 edge and a level-$(n-3)$ edge. Let $s_h^{pq} = \langle h, pq0^{n-2}\rangle$ and $t_h^{pq} = \langle h, pqy\rangle$ for any $p, q \in \mathbb{Z}_2$ and $h \in \{0, 1, 2\}$. As a consequence, $s_0^{00} = s$. Let Γ be the subgraph of $BF(n-2)$ generated by $E(P_1') \cup E(P_2') \cup E(P_3')$. Since $BF_{0,1}^{i,j}(n) = \gamma_1^j \circ \gamma_0^i(BF(n-2))$ and Γ is 0-designed with respect to $BF(n-2)$, $\gamma_1^j \circ \gamma_0^i(\Gamma)$ spans $BF_{0,1}^{i,j}(n)$ and consists of three disjoint paths $\{P_1^{ij}, P_2^{ij}, P_3^{ij}\}$ from s_2^{ij} to d. In particular, let $P_1^{ij} = \langle s_2^{ij}, s_1^{ij}, s_0^{ij}, D_{ij}, d\rangle$ with some D_{ij}. Moreover, P_2^{ij} and P_3^{ij} form a cycle which can be written as $\langle s_2^{ij}, H_{ij}, t_0^{ij}, t_1^{ij}, t_2^{ij}, W_{ij}, d, A_{00}, s_2^{ij}\rangle$. By Lemma 4, there is a totally scheduled hamiltonian cycle $\langle s_0^{pq}, s_1^{pq}, s_2^{pq}, R_{pq}, t_0^{pq}, t_1^{pq}, t_2^{pq}, Q_{pq}, s_0^{pq}\rangle$ of $BF_{0,1}^{p,q}(n)$ for $pq \in \mathbb{Z}_2^2 - \{ij\}$. Then we create a 3*-container $\{P_1, P_2, P_3\}$ of $BF(n)$ as follows.

Subcase 2.1: If $ij = 00$, then $P_1 = D_{00}$, $P_2 = \langle s, s_1^{00}, s_0^{10}, Q_{10}^{-1}, t_2^{10}, t_1^{10}, t_0^{00}, H_{00}^{-1}, s_2^{00}, A_{00}^{-1}, d\rangle$, and $P_3 = \langle s, s_1^{10}, s_2^{11}, R_{11}, t_0^{11}, t_1^{11}, t_2^{11}, Q_{11}, s_0^{11}, s_1^{01}, s_2^{01}, R_{01}, t_0^{01}, t_1^{01}, t_2^{01}, Q_{01}, s_0^{01}, s_1^{11}, s_2^{10}, R_{10}, t_0^{10}, t_1^{00}, t_2^{00}, W_{00}, d\rangle$. Obviously, $s_0^{11} \in I(P_3)$ and $\mu_{s,d}(s_0^{11}) = d_0^{11} \in I(P_3)$. See Figure 8(a).

Subcase 2.2: If $ij = 10$, then $P_1 = \langle s, Q_{00}^{-1}, t_2^{00}, t_1^{00}, t_0^{10}, H_{10}^{-1}, s_2^{10}, A_{10}^{-1}, d\rangle$, $P_2 = \langle s, s_1^{10}, s_0^{10}, D_{10}, d\rangle$, and $P_3 = \langle s, s_1^{00}, s_2^{01}, R_{01}, t_0^{01}, t_1^{01}, t_2^{01}, Q_{01}, s_0^{01}, s_1^{11}, s_2^{11}, R_{11}, t_0^{11}, t_1^{11}, t_2^{11}, Q_{11}, s_0^{11}, s_1^{01}, s_2^{00}, R_{00}, t_0^{00}, t_1^{10}, t_2^{10}, W_{10}, d\rangle$. Obviously, $s_0^{01} \in I(P_3)$ and $\mu_{s,d}(s_0^{01}) = d_0^{11} \in I(P_3)$. See Figure 8(b).

Subcase 2.3: If $ij = 01$, then $P_1 = \langle s, Q_{00}^{-1}, t_2^{00}, t_1^{01}, t_0^{01}, H_{01}^{-1}, s_2^{01}, A_{01}^{-1}, d\rangle$, $P_2 = \langle s, s_1^{00}, s_2^{00}, R_{00}, t_0^{00}, t_1^{00}, t_2^{01}, W_{01}, d\rangle$, and $P_3 = \langle s, s_1^{10}, s_0^{10}, Q_{10}^{-1}, t_2^{10}, t_1^{10}, t_0^{10}, R_{10}^{-1}, s_2^{10}, s_1^{11}, s_2^{11}, R_{11}, t_0^{11}, t_1^{11}, t_2^{11}, Q_{11}, s_0^{11}, s_1^{01}, s_0^{01}, D_{01}, d\rangle$. Obviously, $s_0^{10} \in I(P_3)$ and $\mu_{s,d}(s_0^{10}) = d_0^{11} \in I(P_3)$. See Figure 8(c).

Subcase 2.4: If $ij = 11$, then $P_1 = \langle s, Q_{00}^{-1}, t_2^{00}, t_1^{01}, t_0^{11}, H_{11}^{-1}, s_2^{11}, A_{11}^{-1}, d\rangle$, $P_2 = \langle s, s_1^{00}, s_2^{00}, R_{00}, t_0^{00}, t_1^{00}, t_2^{00}, Q_{01}, s_0^{01}, s_1^{01}, s_2^{01}, R_{01}, t_0^{01}, t_1^{11}, t_2^{11}, W_{11}, d\rangle$, and $P_3 = \langle s, s_1^{10}, s_0^{10}, Q_{10}^{-1}, t_2^{10}, t_1^{10}, t_0^{10}, R_{10}^{-1}, s_2^{10}, s_1^{11}, s_0^{11}, D_{11}, d\rangle$. Obviously, $s_0^{10} \in I(P_3)$ and $\mu_{s,d}(s_0^{10}) = d_0^{01} \in I(P_2)$. See Figure 8(d). □

B Proof of Lemma 10

Proof. Without loss of generality, we assume $\ell_s = 1$ and $\ell_d = 2$ so that $s = \langle 1, 0^n\rangle$ and $d = \langle 2, ijx\rangle$ with some $i, j \in \mathbb{Z}_2$, and $x \in \mathbb{Z}_2^{n-2}$. To construct the desired 3*-container of $BF(n)$, we partition $BF(n)$ into $\{BF_{0,1}^{p,q}(n) \mid p, q \in \mathbb{Z}_2\}$.

Case 1: Suppose that $x = 0^{n-2}$. Assume that $w = \langle 1, 00y\rangle$ is a level-1 vertex of $BF_{0,1}^{0,0}(n)$ other than s. Let $u_h^{pq} = \langle h, pq0^{n-2}\rangle$ and $w_h^{pq} = \langle h, pqy\rangle$ for any $p, q \in \mathbb{Z}_2$ and $h \in \{0, 1, 2\}$. Thus, $w_1^{00} = w$, $u_1^{00} = s$, and $u_2^{ij} = d$. By Lemma 4, there is a totally scheduled hamiltonian cycle $C^{pq} = \langle u_1^{pq}, u_2^{pq}, R_{pq}, w_0^{pq}, w_1^{pq}, w_2^{pq}, D_{pq}, u_0^{pq}, u_1^{pq}\rangle$ of $BF_{0,1}^{p,q}(n)$ for any $p, q \in \mathbb{Z}_2$. Then we build a 3*-container $\{P_1, P_2, P_3\}$ of $BF(n)$ joining s to d as follows.

Subcase 1.1: If $ij = 00$, then $P_1 = \langle s, d\rangle$, $P_2 = \langle s, u_0^{00}, D_{00}^{-1}, w_2^{00}, w_1^{00}, w_0^{00}, R_{00}^{-1}, d\rangle$, and $P_3 = \langle s, u_2^{01}, R_{01}, w_0^{01}, w_1^{01}, w_2^{01}, D_{01}, u_0^{01}, u_1^{11}, u_2^{10}, R_{10}, w_0^{10}, w_1^{10}, w_2^{10}, D_{10}, u_0^{10}, u_1^{10}, u_2^{11}, R_{11}, w_0^{11}, w_1^{11}, w_2^{11}, D_{11}, u_0^{11}, u_1^{01}, d\rangle$. See Figure 9(a) for illustration.

Subcase 1.2: If $ij = 10$, then $P_1 = \langle s, u_2^{01}, R_{01}, w_0^{01}, w_1^{01}, w_2^{01}, D_{01}, u_0^{01}, u_1^{01}, u_0^{11}, D_{11}^{-1}, w_2^{11}, w_1^{11}, w_0^{11}, R_{11}^{-1}, u_2^{11}, u_1^{11}, d\rangle$, $P_2 = \langle s, u_2^{00}, R_{00}, w_0^{00}, w_1^{10}, w_2^{10}, D_{10}, u_0^{10}, u_1^{10}, d\rangle$, and $P_3 = \langle s, u_0^{00}, D_{00}^{-1}, w_2^{00}, w_1^{00}, w_0^{10}, R_{10}^{-1}, d\rangle$. See Figure 9(b).

Subcase 1.3: If $ij = 01$, then $P_1 = \langle s, u_0^{00}, D_{00}^{-1}, w_2^{00}, w_1^{01}, w_0^{11}, R_{11}^{-1}, u_2^{11}, u_1^{11}, u_0^{11}, D_{11}^{-1}, w_2^{11}, w_1^{11}, w_0^{01}, R_{01}^{-1}, d\rangle$, $P_2 = \langle s, d\rangle$, and $P_3 = \langle s, u_2^{00}, R_{00}, w_0^{00}, w_1^{10}, w_2^{10}, D_{10}, u_0^{10}, u_1^{10}, u_2^{10}, R_{10}, w_0^{10}, w_1^{00}, w_2^{01}, D_{01}, u_0^{01}, u_1^{01}, d\rangle$. See Figure 9(c).

Subcase 1.4: If $ij = 11$, then $P_1 = \langle s, u_2^{01}, R_{01}, w_0^{01}, w_1^{01}, w_2^{01}, D_{01}, u_0^{01}, u_1^{01}, u_0^{11}, D_{11}^{-1}, w_2^{11}, w_1^{11}, w_0^{11}, R_{11}^{-1}, d\rangle$, $P_2 = \langle s, u_2^{00}, R_{00}, w_0^{00}, w_1^{00}, w_2^{00}, D_{00}, u_0^{00}, u_1^{10}, d\rangle$, and $P_3 = \langle s, u_0^{10}, D_{10}^{-1}, w_2^{10}, w_1^{10}, w_0^{10}, R_{10}^{-1}, u_2^{10}, u_1^{11}, d\rangle$. See Figure 9(d) for illustration.

Case 2: Suppose that $x \neq 0^{n-2}$. Let $s_h^{pq} = \langle h, pq0^{n-2}\rangle$ and $d_h^{pq} = \langle h, pqx\rangle$ for any $p, q \in \mathbb{Z}_2, h \in \{0, 1, 2\}$. Thus, $s_1^{00} = s$ and $d_2^{ij} = d$. By Lemma 4, there is a totally scheduled hamiltonian cycle $C^{pq} = \langle s_1^{pq}, s_2^{pq}, R_{pq}, d_0^{pq}, d_1^{pq}, d_2^{pq}, D_{pq}, s_0^{pq}, s_1^{pq}\rangle$ of $BF_{0,1}^{p,q}(n)$ for any $p, q \in \mathbb{Z}_2$. By Corollary 2, there is a 2-scheduled hamiltonian path $\langle s_0^{hk}, T_{hk}, d_1^{hk}\rangle$ of $BF_{0,1}^{h,k}(n)$ for $hk \in \{10, 11\}$. Then we create a 3*-container $\{P_1, P_2, P_3\}$ of $BF(n)$ joining s to d, as illustrated in Figure 10.

Subcase 2.1: If $ij = 00$, then $P_1 = \langle s, s_2^{00}, R_{00}, d_0^{00}, d_1^{00}, d\rangle$, $P_2 = \langle s, s_0^{00}, D_{00}^{-1}, d\rangle$, and $P_3 = \langle s, s_0^{10}, T_{10}, d_1^{10}, d_2^{11}, D_{11}, s_0^{11}, s_1^{11}, s_2^{11}, R_{11}, d_0^{11}, d_1^{11}, d_0^{01}, R_{01}^{-1}, s_2^{01}, s_1^{01}, s_0^{01}, D_{01}^{-1}, d_2^{01}, d_1^{01}, d\rangle$.

Subcase 2.2: If $ij = 10$, then $P_1 = \langle s, s_2^{00}, R_{00}, d_0^{00}, d_1^{10}, d\rangle$, $P_2 = \langle s, s_0^{00}, D_{00}^{-1}, d_2^{00}, d_1^{00}, d_0^{10}, R_{10}^{-1}, s_2^{10}, s_1^{10}, s_0^{10}, D_{10}^{-1}, d\rangle$, and $P_3 = \langle s, s_2^{01}, R_{01}, d_0^{01}, d_1^{01}, d_2^{01}, D_{01}, s_0^{01}, s_1^{01}, s_0^{11}, T_{11}, d_1^{11}, d\rangle$.

Subcase 2.3: If $ij = 01$, then $P_1 = \langle s, s_2^{00}, R_{00}, d_0^{00}, d_1^{00}, d\rangle$, $P_2 = \langle s, s_0^{00}, D_{00}^{-1}, d_2^{00}, d_1^{01}, d\rangle$, and $P_3 = \langle s, s_0^{10}, T_{10}, d_1^{10}, d_2^{11}, D_{11}, s_0^{11}, s_1^{11}, s_2^{11}, R_{11}, d_0^{11}, d_1^{11}, d_0^{01}, R_{01}^{-1}, s_2^{01}, s_1^{01}, s_0^{01}, D_{01}^{-1}, d\rangle$.

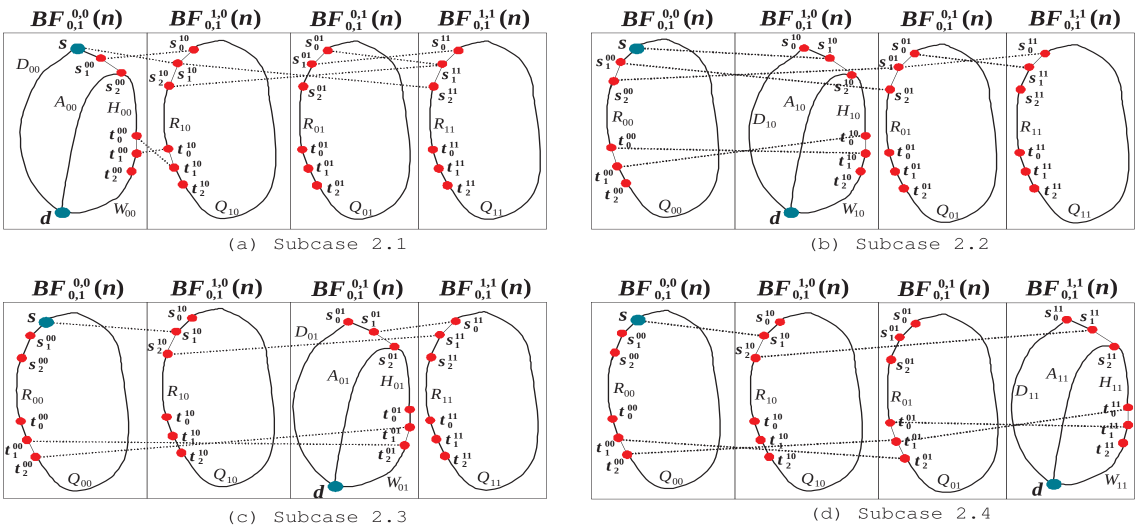


Figure 8: Illustrations for Case 2 of Lemma 9.

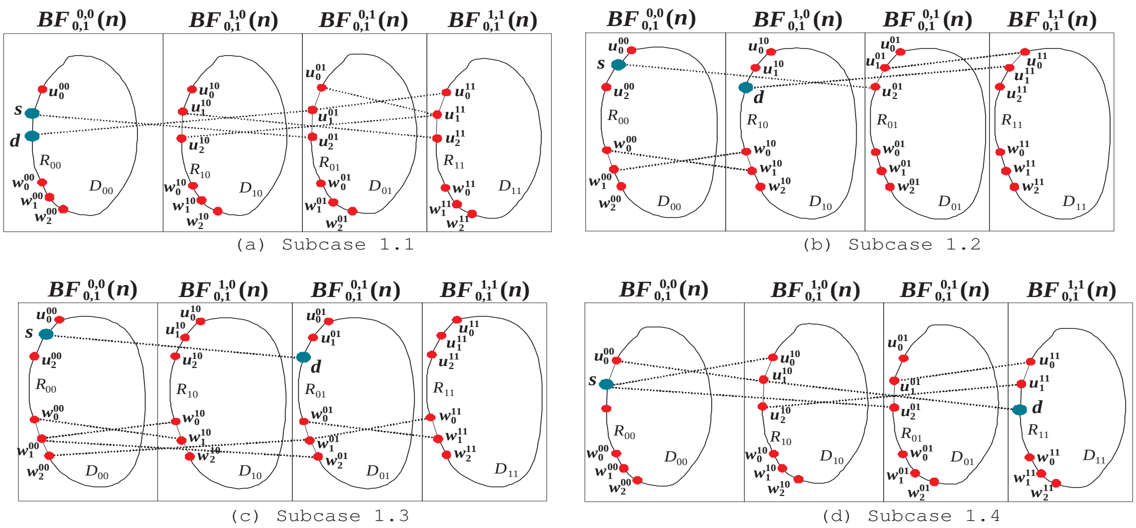


Figure 9: Illustrations for Case 1 of Lemma 10.

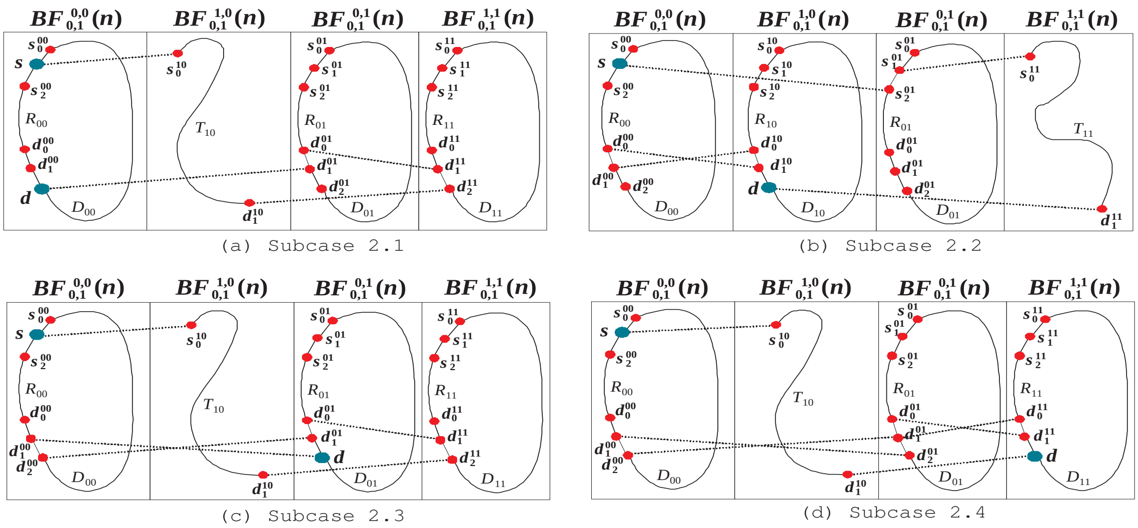


Figure 10: Illustrations for Case 2 of Lemma 10.

Subcase 2.4: If $ij = 11$, then $P_1 = \langle s, s_2^{00}, R_{00}, d_0^{00}, d_1^{00}, d_2^{01}, D_{01}, s_0^{01}, s_1^{01}, s_2^{01}, R_{01}, d_0^{01}, d_1^{11}, d\rangle$, $P_2 = \langle s, s_0^{00}, D_{00}^{-1}, d_2^{00}, d_1^{01}, d_0^{11}, R_{11}^{-1}, s_2^{11}, s_1^{11}, s_0^{11}, D_{11}^{-1}, d\rangle$, and $P_3 = \langle s, s_0^{10}, T_{10}, d_1^{10}, d\rangle$. □

C Proof of Lemma 11

Proof. Without loss of generality, we assume $s = \langle 0, 0000\rangle$ and $d = \langle 3, aijb\rangle$ with some $i, j, a, b \in \mathbb{Z}_2$. To create the desired 3^*-container, we may partition $BF(4)$ into $\{BF_{1,2}^{p,q}(4) \mid p, q \in \mathbb{Z}_2\}$. Let $d_h^{pq} = \langle 1 + h, apqb\rangle$ for any $p, q \in \mathbb{Z}_2$, $h \in \{0, 1, 2\}$. By brute force, we design three disjoint paths $\{P_1^{00}, P_2^{00}, P_3^{00}\}$ of $BF_{1,2}^{0,0}(4)$ joining s to d_0^{00} such that $V(P_1^{00}) \cup V(P_2^{00}) \cup V(P_3^{00}) = V(BF_{1,2}^{0,0}(4))$. See Table 5 for the details. Accordingly, we can write P_1^{00} as $\langle s, W_{00}, d_2^{00}, d_1^{00}, d_0^{00}\rangle$. Moreover, P_2^{00} and P_3^{00} form a cycle that can be written as $\langle s, R_{00}, t_0^{00}, t_1^{00}, t_2^{00}, Q_{00}, d_0^{00}, A_{00}, s\rangle$, in which $t_h^{00} = \langle h + 1, y_1 00 y_2\rangle$ with some $y_1y_2 \neq ab$. Let $t_h^{pq} = \langle 1 + h, y_1pqy_2\rangle$. By Lemma 4, there is a totally scheduled hamiltonian cycle $C^{pq} = \langle t_0^{pq}, t_1^{pq}, t_2^{pq}, R_{pq}, d_0^{pq}, d_1^{pq}, d_2^{pq}, Q_{pq}, t_0^{pq}\rangle$ of $BF_{1,2}^{p,q}(4)$ for any $pq \in \mathbb{Z}_2^2 - \{00\}$. Then we list the 3^*-container $\{P_1, P_2, P_3\}$ of $BF(4)$ joining s to d in Table 6. □

D Proof of Lemma 12

Proof. Without loss of generality, we assume that $s = \langle 0, 0^n\rangle$ and $d = \langle \ell_d, x_1ijx_2\rangle$ with some $3 \le \ell_d \le n-1$, $i, j \in \mathbb{Z}_2$, $x_1 \in \mathbb{Z}_2^{\ell_d-2}$, and $x_2 \in \mathbb{Z}_2^{n-\ell_d}$. Then we partition $BF(n)$ into $\{BF_{\ell_d-2,\ell_d-1}^{p,q}(n) \mid p, q \in \mathbb{Z}_2\}$ to create the desired 3^*-container by induction on n.

The induction bases depend upon both Lemma 10 and Lemma 11. Next, we assume that the statement holds for $BF(n-2)$ with $n \ge 5$; that is, there exists a 3^*-container $\{P_1', P_2', P_3'\}$ of $BF(n-2)$ joining $s' = \langle 0, 0^{n-2}\rangle$ to $d' = \langle \ell_d - 2, x_1x_2\rangle$ such that P_1' ends up with a level-$(\ell_d - 2)$ edge and also that there is a level-$(\ell_d - 2)$ vertex t of $I(P_3')$ incident with a level-$(\ell_d - 2)$ edge and a level-$(\ell_d - 3)$ edge. We set $t = \langle \ell_d - 2, y_1y_2\rangle$ with some $y_1 \in \mathbb{Z}_2^{\ell_d-2}$, $y_2 \in \mathbb{Z}_2^{n-\ell_d}$, and $y_1y_2 \neq x_1x_2$. Let Γ be the subgraph of $BF(n-2)$ generated by $E(P_1') \cup E(P_2') \cup E(P_3')$. Let $d_h^{pq} = \langle \ell_d - 2 + h, x_1pqx_2\rangle$ and $t_h^{pq} = \langle \ell_d - 2 + h, y_1pqy_2\rangle$ for any $p, q \in \mathbb{Z}_2$ and $h \in \{0, 1, 2\}$. Since $BF_{\ell_d-2,\ell_d-1}^{0,0}(n) = \gamma_{\ell_d-1}^0 \circ \gamma_{\ell_d-2}^0(BF(n-2))$ and Γ is $(\ell_d - 2)$-designed, $\gamma_{\ell_d-1}^0 \circ \gamma_{\ell_d-2}^0(\Gamma)$ spans $BF_{\ell_d-2,\ell_d-1}^{0,0}(n)$ and consists of three disjoint paths $\{H_1, H_2, H_3\}$ between s and d_0^{00}. Suppose H_1 ends up with a level-$(\ell_d - 2)$ edge. Accordingly, we have $H_1 = \langle s, \gamma_{\ell_d-1}^0 \circ \gamma_{\ell_d-2}^0(P_1') = W_{00}, d_2^{00}, d_1^{00}, d_0^{00}\rangle$. Moreover, H_2 and H_3 form a cycle which can be written as $\langle s, R_{00}, t_0^{00}, t_1^{00}, t_2^{00}, Q_{00}, d_0^{00}, A_{00}, s\rangle$. By Lemma 4, there is a totally scheduled hamiltonian cycle $C^{pq} = \langle t_0^{pq}, t_1^{pq}, t_2^{pq}, R_{pq}, d_0^{pq}, d_1^{pq}, d_2^{pq}, Q_{pq}, t_0^{pq}\rangle$ of $BF_{\ell_d-2,\ell_d-1}^{p,q}(n)$ for $pq \in \mathbb{Z}_2^2 - \{00\}$. Then we also list the 3^*-container $\{P_1, P_2, P_3\}$ of $BF(n)$ joining s to d in Table 6. □

E Proof of Proposition 1

Proof. Without loss of generality, we assume that $s = \langle 0, 0^n\rangle$ and $d = \langle 0, ijx\rangle$ with some $i, j \in \mathbb{Z}_2$ and $x \in \mathbb{Z}_2^{n-2}$. We prove this lemma by induction on n. The induction bases are listed in Table 7. Next, we suppose that the statement holds for $BF(n-2)$ with $n \ge 5$. Then we partition $BF(n)$ into $\{BF_{0,1}^{p,q}(n) \mid p, q \in \mathbb{Z}_2\}$ to create the desired container.

Case 1: Suppose that $x = 0^{n-2}$. Let $s' = \langle 0, 0^{n-2}\rangle$ and $t = \langle 0, z\rangle$ be two distinct level-0 vertices of $BF(n-2)$. By Corollary 3, there is a 3^*-container $\{Q_1, Q_2, Q_3\}$ of $BF(n-2)$ joining s' to t such that Q_3 ends up with a level-0 edge and also that there is a level-0 vertex $u = \langle 0, y\rangle$ of $I(Q_3)$ incident with a level-0 edge and a level-$(n-3)$ edge. Let $t_h^{pq} = \langle h, pqz\rangle$ and $u_h^{pq} = \langle h, pqy\rangle$ for any $p, q \in \mathbb{Z}_2$, $h \in \{0, 1, 2\}$. Furthermore, let Γ be the 0-designed subgraph of $BF(n-2)$ generated by $E(Q_1) \cup E(Q_2) \cup E(Q_3)$. Since $BF_{0,1}^{p,q}(n) = \gamma_1^q \circ \gamma_0^p(BF(n-2))$, $\gamma_1^q \circ \gamma_0^p(\Gamma)$ spans $BF_{0,1}^{p,q}(n)$ and consists of three disjoint paths $\{P_1^{pq}, P_2^{pq}, P_3^{pq}\}$. In particular, suppose that P_1^{pq} ends up with a level-1 edge. Accordingly, P_2^{pq} and P_3^{pq} form a cycle containing u_0^{pq}, u_1^{pq}, and u_2^{pq}. By Lemma 4, there is a totally scheduled hamiltonian cycle $C^{pq} = \langle t_1^{pq}, t_2^{pq}, R_{pq}, t_0^{pq}, t_1^{pq}\rangle$ or $O^{pq} = \langle u_1^{pq}, u_2^{pq}, T_{pq}, t_0^{pq}, u_1^{pq}\rangle$ of $BF_{0,1}^{p,q}(n)$ for any $p, q \in \mathbb{Z}_2$. By Lemma 7, there is a hamiltonian cycle $H^{pq} = \langle u_0^{pq}, u_1^{pq}, u_2^{pq}, W_{pq}, u_0^{pq}\rangle$ of $BF_{0,1}^{p,q}(n) - \{t_0^{pq}, t_1^{pq}, t_2^{pq}\}$.

Subcase 1.1: Assume that $ij = 10$. Let $A = \{(t_1^{00}, t_2^{01}), (t_0^{01}, t_1^{11}), (t_2^{11}, t_1^{10}), (u_2^{00}, u_1^{01}), (u_2^{01}, u_1^{00}), (u_1^{00}, u_0^{10}), (u_0^{00}, u_1^{10})\}$ and let $B = \{(t_1^{00}, t_2^{00}), (u_0^{00}, u_1^{00}), (u_1^{00}, u_2^{00}), (u_0^{10}, u_1^{10}), (t_1^{10}, t_2^{10}), (u_1^{01}, u_2^{01}), (t_1^{11}, t_2^{11})\}$. Then the subgraph generated by $(E(\gamma_1^0 \circ \gamma_0^0(\Gamma)) \cup E(\gamma_1^0 \circ \gamma_0^1(\Gamma)) \cup E(H^{01}) \cup E(C^{11}) \cup A) - B$ forms a 3^*-container. See Figure 12(a) for illustration.

Subcase 1.2: Assume that $ij = 01$. Let $A = \{(t_1^{00}, t_0^{10}), (t_2^{10}, t_1^{11}), (t_0^{11}, t_1^{01}), (u_2^{00}, u_1^{01}), (u_0^{00}, u_1^{10}), (u_0^{10}, u_1^{00}), (u_1^{00}, u_2^{01})\}$ and let $B = \{(t_1^{00}, t_2^{00}), (u_0^{00}, u_1^{00}), (u_1^{00}, u_2^{00}), (u_0^{10}, u_1^{10}), (t_1^{01}, t_2^{01}), (u_1^{01}, u_2^{01}), (t_0^{11}, t_1^{11})\}$. Then the subgraph generated by $(E(\gamma_1^0 \circ \gamma_0^0(\Gamma)) \cup E(\gamma_1^1 \circ \gamma_0^0(\Gamma)) \cup E(H^{10}) \cup E(C^{11}) \cup A) - B$ forms a 3^*-container. See Figure 12(b) for illustration.

Subcase 1.3: Assume that $ij = 11$. Let $A = \{(t_1^{00}, t_2^{01}), (t_0^{01}, t_1^{11}), (u_2^{00}, u_1^{01}), (u_2^{01}, u_1^{00}), (u_1^{00}, u_0^{10}), (u_2^{10}, u_1^{11}), (u_0^{00}, u_1^{10}), (u_1^{10}, u_2^{11})\}$ and let $B = \{(t_1^{00}, t_2^{00}), (u_0^{00}, u_1^{00}), (u_1^{00}, u_2^{00}), (u_0^{10}, u_1^{10}), (u_1^{10}, u_2^{10}), (u_1^{01}, u_2^{01}), (t_1^{11}, t_2^{11}), (u_1^{11}, u_2^{11})\}$. Then the subgraph generated by $(E(\gamma_1^0 \circ \gamma_0^0(\Gamma)) \cup E(\gamma_1^1 \circ \gamma_0^1(\Gamma)) \cup E(H^{01}) \cup E(O^{10}) \cup A) - B$ forms a 3^*-container. See Figure 12(c).

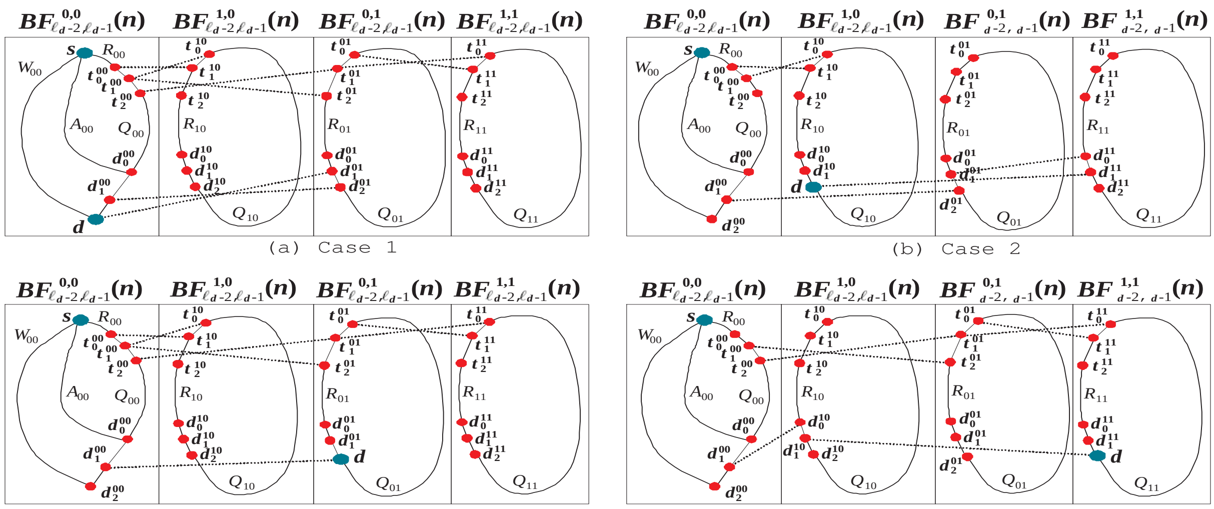
Case 2: Suppose that $x \neq 0^{n-2}$. We distinguish the following subcases.

Table 5: Sets of three disjoint paths of $BF^{0,0}_{1,2}(4)$. The vertices t^{00}_0, t^{00}_1, and t^{00}_2 are marked by star symbols.

w	Sets of three disjoint paths $\{P^{00}_1, P^{00}_2, P^{00}_3\}$ from s to d^{00}_0
$d^{00}_0 = \langle 1, 0000\rangle$	$P^{00}_1 = \langle\langle 0,0000\rangle, \langle 3,0001\rangle, \langle 2,0001\rangle, \langle 1,0001\rangle, \langle 0,1001\rangle, \langle 3,1001\rangle, \langle 2,1001\rangle, \langle 1,1001\rangle, \langle 0,0001\rangle, \langle 3,0000\rangle, \langle 2,0000\rangle, \langle 1,0000\rangle\rangle$
	$P^{00}_2 = \langle\langle 0,0000\rangle, \langle 1,0000\rangle\rangle$
	$P^{00}_3 = \langle\langle 0,0000\rangle, \langle 1,1000\rangle^*, \langle 2,1000\rangle^*, \langle 3,1000\rangle^*, \langle 0,1000\rangle, \langle 1,0000\rangle\rangle$
$d^{00}_0 = \langle 1, 1000\rangle$	$P^{00}_1 = \langle\langle 0,0000\rangle, \langle 3,0001\rangle, \langle 2,0001\rangle, \langle 1,0001\rangle, \langle 0,1001\rangle, \langle 3,1000\rangle, \langle 2,1000\rangle, \langle 1,1000\rangle\rangle$
	$P^{00}_2 = \langle\langle 0,0000\rangle, \langle 1,1000\rangle\rangle$
	$P^{00}_3 = \langle\langle 0,0000\rangle, \langle 1,0000\rangle, \langle 2,0000\rangle, \langle 3,0000\rangle, \langle 0,0001\rangle, \langle 1,1001\rangle^*, \langle 2,1001\rangle^*, \langle 3,1001\rangle^*, \langle 0,1000\rangle, \langle 1,1000\rangle\rangle$
$d^{00}_0 = \langle 1, 0001\rangle$	$P^{00}_1 = \langle\langle 0,0000\rangle, \langle 3,0001\rangle, \langle 2,0001\rangle, \langle 1,0001\rangle\rangle$
	$P^{00}_2 = \langle\langle 0,0000\rangle, \langle 1,0000\rangle, \langle 2,0000\rangle, \langle 3,0000\rangle, \langle 0,0001\rangle, \langle 1,0001\rangle\rangle$
	$P^{00}_3 = \langle\langle 0,0000\rangle, \langle 1,1000\rangle^*, \langle 2,1000\rangle^*, \langle 3,1000\rangle^*, \langle 0,1000\rangle, \langle 3,1001\rangle, \langle 2,1001\rangle, \langle 1,1001\rangle, \langle 0,1001\rangle, \langle 1,0001\rangle\rangle$
$d^{00}_0 = \langle 1, 1001\rangle$	$P^{00}_1 = \langle\langle 0,0000\rangle, \langle 1,1000\rangle, \langle 2,1000\rangle, \langle 3,1000\rangle, \langle 0,1000\rangle, \langle 3,1001\rangle, \langle 2,1001\rangle, \langle 1,1001\rangle\rangle$
	$P^{00}_2 = \langle\langle 0,0000\rangle, \langle 3,0001\rangle, \langle 2,0001\rangle, \langle 1,0001\rangle, \langle 0,1001\rangle, \langle 1,1001\rangle\rangle$
	$P^{00}_3 = \langle\langle 0,0000\rangle, \langle 1,0000\rangle^*, \langle 2,0000\rangle^*, \langle 3,0000\rangle^*, \langle 0,0001\rangle, \langle 1,1001\rangle\rangle$

Table 6: 3^*-container $\{P_1, P_2, P_3\}$ of $BF(n)$ joining s to d for Lemma 11 and for Lemma 12, with reference to Figure 11.

Case 1: $ij = 00$	$P_1 = W_{00}$
	$P_2 = \langle s, A^{-1}_{00}, d^{00}_0, Q^{-1}_{00}, t^{00}_2, t^{01}_1, t^{11}_0, Q^{-1}_{11}, d^{11}_2, d^{11}_1, d^{11}_0, R^{-1}_{11}, t^{11}_2, t^{11}_1, t^{01}_0, Q^{-1}_{01}, d^{01}_2, d^{00}_1, d\rangle$
	$P_3 = \langle s, R_{00}, t^{00}_0, t^{10}_1, t^{10}_2, R_{10}, d^{10}_0, d^{10}_1, d^{10}_2, Q_{10}, t^{10}_0, t^{00}_1, t^{01}_2, R_{01}, d^{01}_0, d^{01}_1, d\rangle$
Case 2: $ij = 10$	$P_1 = \langle s, A^{-1}_{00}, d^{00}_0, Q^{-1}_{00}, t^{00}_2, t^{00}_1, t^{10}_0, Q^{-1}_{10}, d\rangle$
	$P_2 = \langle s, R_{00}, t^{00}_0, t^{10}_1, t^{10}_2, R_{10}, d^{10}_0, d^{10}_1, d\rangle$
	$P_3 = \langle s, W_{00}, d^{00}_2, d^{00}_1, d^{01}_2, Q_{01}, t^{01}_0, t^{01}_1, t^{01}_2, R_{01}, d^{01}_0, d^{01}_1, d^{11}_0, R^{-1}_{11}, t^{11}_2, t^{11}_1, t^{11}_0, Q^{-1}_{11}, d^{11}_2, d^{11}_1, d\rangle$
Case 3: $ij = 01$	$P_1 = \langle s, A^{-1}_{00}, d^{00}_0, Q^{-1}_{00}, t^{00}_2, t^{01}_1, t^{11}_0, Q^{-1}_{11}, d^{11}_2, d^{11}_1, d^{11}_0, R^{-1}_{11}, t^{11}_2, t^{11}_1, t^{01}_0, Q^{-1}_{01}, d\rangle$
	$P_2 = \langle s, W_{00}, d^{00}_2, d^{00}_1, d\rangle$
	$P_3 = \langle s, R_{00}, t^{00}_0, t^{10}_1, t^{10}_2, R_{10}, d^{10}_0, d^{10}_1, d^{10}_2, Q_{10}, t^{10}_0, t^{00}_1, t^{01}_2, R_{01}, d^{01}_0, d^{01}_1, d\rangle$
Case 4: $ij = 11$	$P_1 = \langle s, A^{-1}_{00}, d^{00}_0, Q^{-1}_{00}, t^{00}_2, t^{01}_1, t^{11}_0, Q^{-1}_{11}, d\rangle$
	$P_2 = \langle s, W_{00}, d^{00}_2, d^{00}_1, d^{10}_0, R^{-1}_{10}, t^{10}_2, t^{10}_1, t^{10}_0, Q^{-1}_{10}, d^{10}_2, d^{10}_1, d\rangle$
	$P_3 = \langle s, R_{00}, t^{00}_0, t^{00}_1, t^{01}_2, R_{01}, d^{01}_0, d^{01}_1, d^{01}_2, Q_{01}, t^{01}_0, t^{11}_1, t^{11}_2, R_{11}, d^{11}_0, d^{11}_1, d\rangle$



(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4

Figure 11: Illustrations for Lemma 12 (also for Lemma 11 when $\ell_d = 3$).

Table 7: 3^*-containers of $BF(3)$ as induction bases.

d	3^*-container $\{J_1, J_2, J_3\}$ joining $\langle 0, 000\rangle$ to d
$\langle 0, 100\rangle$	$J_1 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 2,110\rangle, \langle 0,111\rangle, \langle 1,111\rangle, \langle 2,111\rangle, \langle 0,110\rangle, \langle 1,110\rangle, \langle 2,100\rangle, \langle 0,100\rangle\rangle$
	$J_2 = \langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 1,010\rangle, \langle 0,010\rangle, \langle 2,011\rangle, \langle 1,011\rangle, \langle 0,011\rangle, \langle 2,010\rangle, \langle 1,000\rangle, \langle 0,100\rangle\rangle$
	$J_3 = \langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 1,001\rangle, \langle 0,001\rangle, \langle 1,101\rangle, \langle 0,101\rangle, \langle 2,101\rangle, \langle 0,100\rangle\rangle$
$\langle 0, 010\rangle$	$J_1 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 2,100\rangle, \langle 0,100\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 0,010\rangle\rangle$
	$J_2 = \langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 1,010\rangle, \langle 0,110\rangle, \langle 2,110\rangle, \langle 1,110\rangle, \langle 0,010\rangle\rangle$
	$J_3 = \langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 1,001\rangle, \langle 0,001\rangle, \langle 1,101\rangle, \langle 0,101\rangle, \langle 2,101\rangle, \langle 1,111\rangle, \langle 2,111\rangle, \langle 0,111\rangle, \langle 1,011\rangle, \langle 0,011\rangle, \langle 2,011\rangle, \langle 0,010\rangle\rangle$
$\langle 0, 110\rangle$	$J_1 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 0,100\rangle, \langle 2,101\rangle, \langle 1,101\rangle, \langle 0,101\rangle, \langle 2,100\rangle, \langle 1,110\rangle, \langle 2,110\rangle, \langle 0,110\rangle\rangle$
	$J_2 = \langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 0,010\rangle, \langle 1,010\rangle, \langle 0,110\rangle\rangle$
	$J_3 = \langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 2,011\rangle, \langle 1,011\rangle, \langle 0,011\rangle, \langle 1,111\rangle, \langle 0,111\rangle, \langle 2,111\rangle, \langle 0,110\rangle\rangle$
$\langle 0, 001\rangle$	$J_1 = \langle\langle 0,000\rangle, \langle 1,000\rangle, \langle 0,100\rangle, \langle 1,100\rangle, \langle 2,110\rangle, \langle 0,111\rangle, \langle 1,011\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,111\rangle, \langle 2,101\rangle, \langle 1,101\rangle, \langle 2,111\rangle, \langle 0,110\rangle, \langle 1,010\rangle,$ $\langle 2,010\rangle, \langle 0,010\rangle, \langle 1,110\rangle, \langle 2,100\rangle, \langle 0,101\rangle, \langle 1,001\rangle, \langle 0,001\rangle\rangle$
	$J_2 = \langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 0,001\rangle\rangle$
	$J_3 = \langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 0,001\rangle\rangle$
$\langle 0, 101\rangle$	$J_1 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 0,100\rangle, \langle 2,100\rangle, \langle 0,101\rangle\rangle$
	$J_2 = \langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 0,001\rangle, \langle 1,001\rangle, \langle 0,101\rangle\rangle$
	$J_3 = \langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 1,010\rangle, \langle 0,110\rangle, \langle 2,110\rangle, \langle 1,110\rangle, \langle 0,010\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,011\rangle, \langle 0,111\rangle, \langle 1,111\rangle, \langle 2,111\rangle,$ $\langle 1,101\rangle, \langle 2,101\rangle, \langle 0,101\rangle\rangle$
$\langle 0, 011\rangle$	$J_1 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 2,100\rangle, \langle 0,100\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 0,011\rangle\rangle$
	$J_2 = \langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 0,001\rangle, \langle 1,101\rangle, \langle 2,101\rangle, \langle 0,101\rangle, \langle 1,001\rangle, \langle 2,011\rangle, \langle 0,011\rangle\rangle$
	$J_3 = \langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 1,011\rangle, \langle 0,111\rangle, \langle 2,110\rangle, \langle 1,110\rangle, \langle 0,010\rangle, \langle 1,010\rangle, \langle 0,110\rangle, \langle 2,111\rangle, \langle 1,111\rangle, \langle 0,011\rangle\rangle$
$\langle 0, 111\rangle$	$J_1 = \langle\langle 0,000\rangle, \langle 1,100\rangle, \langle 0,100\rangle, \langle 2,100\rangle, \langle 1,110\rangle, \langle 0,110\rangle, \langle 2,110\rangle, \langle 0,111\rangle\rangle$
	$J_2 = \langle\langle 0,000\rangle, \langle 2,000\rangle, \langle 1,000\rangle, \langle 2,010\rangle, \langle 1,010\rangle, \langle 0,010\rangle, \langle 2,011\rangle, \langle 0,011\rangle, \langle 1,111\rangle, \langle 2,101\rangle, \langle 0,101\rangle, \langle 1,001\rangle, \langle 0,001\rangle, \langle 1,101\rangle, \langle 2,111\rangle,$ $\langle 0,111\rangle\rangle$
	$J_3 = \langle\langle 0,000\rangle, \langle 2,001\rangle, \langle 1,011\rangle, \langle 0,111\rangle\rangle$

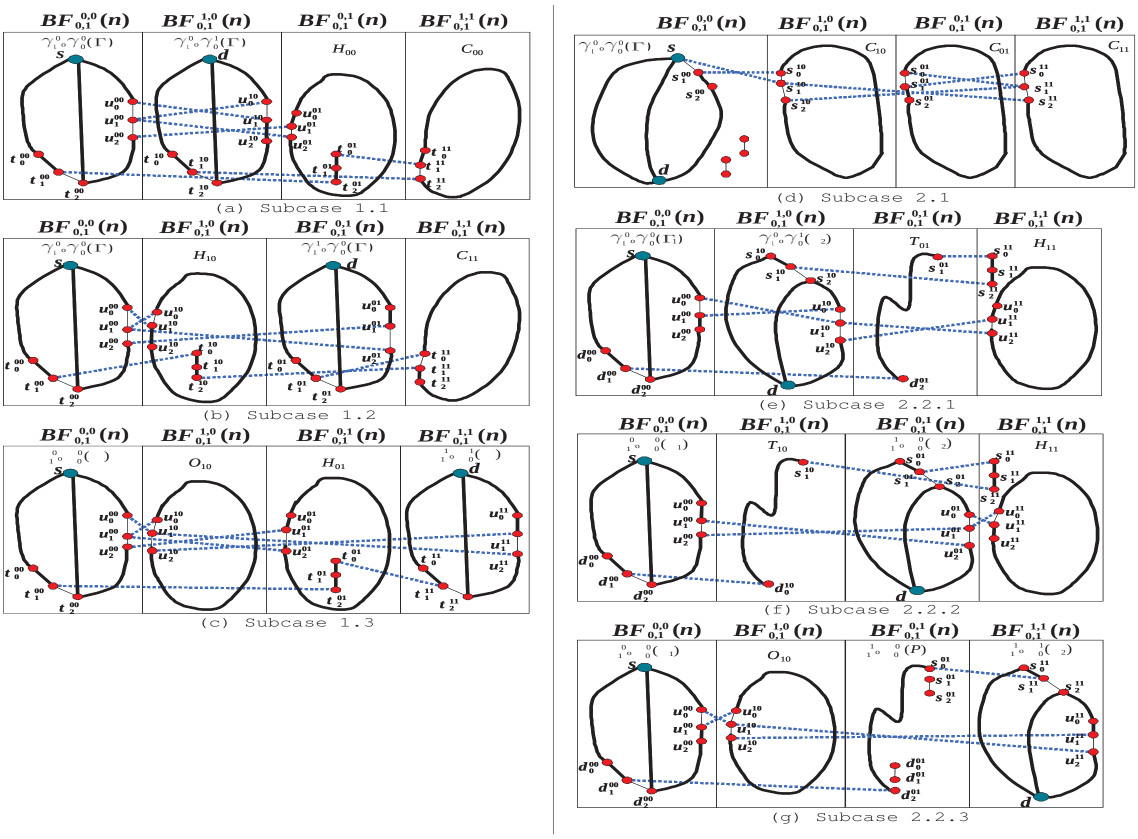


Figure 12: Illustrations for Proposition 1.

Subcase 2.1: If $ij = 00$, $d = \langle 0, 00x\rangle$. By induction hypothesis, there is a 3*-container $\{P_1, P_2, P_3\}$ of $BF(n-2)$ joining $s' = \langle 0, 0^{n-2}\rangle$ to $d' = \langle 0, x\rangle$. Let Γ be the subgraph of $BF(n-2)$ generated by $E(P_1) \cup E(P_2) \cup E(P_3)$. By Lemma 4, there is a totally scheduled hamiltonian cycle $C^{pq} = \langle s_0^{pq}, s_1^{pq}, s_2^{pq}, R_{pq}, s_0^{pq}\rangle$ of $BF_{0,1}^{p,q}(n)$ for any $pq \in \mathbb{Z}_2^2 - \{00\}$. Let $A = \{(s, s_1^{10}),\ (s_1^{10}, s_2^{11}),\ (s_0^{11}, s_1^{01}),\ (s_0^{01}, s_1^{11}),\ (s_1^{11}, s_2^{10}),\ (s_0^{10}, s_1^{00})\}$ and $B = \{(s, s_1^{00}),\ (s_0^{10}, s_1^{10}),\ (s_1^{10}, s_2^{10}),\ (s_0^{01}, s_1^{01}),\ (s_0^{11}, s_1^{11}),\ (s_1^{11}, s_2^{11})\}$. Then the subgraph Ω generated by $\big(E(\gamma_1^0 \circ \gamma_0^0(\Gamma)) \cup E(C^{10}) \cup E(C^{01}) \cup E(C^{11}) \cup A\big) - B$ forms a 3-container of $BF(n)$ between s and d such that each of $\{s, d\}$ is incident with only one level-0 edge. See Figure 12(d). By Lemma 2, $V(BF(n)) - V(\Omega) = V(BF_{0,1}^{0,0}(n)) - V(\gamma_1^0 \circ \gamma_0^0(\Gamma))$. Hence, by Lemma 3, there is a 3*-container of $BF(n)$ between s and d with the desired property.

Subcase 2.2: Suppose that $ij \neq 00$. By Lemma 9, there exists a 3*-container $\{P_1, P_2, P_3\}$ of $BF(n-2)$ joining $s' = \langle 0, 0^{n-2}\rangle$ to $d' = \langle 0, x\rangle$ such that only P_1 begins with a level-$(n-3)$ edge. By Corollary 3, there exists another 3*-container $\{Q_1, Q_2, Q_3\}$ of $BF(n-2)$ joining s' to d' such that only Q_1 ends up with a level-$(n-3)$ edge. Besides, $(I(P_2) \cup I(P_3)) \cap (I(Q_2) \cup I(Q_3))$ contains at least one level-0 vertex u incident with a level-0 edge and a level-$(n-3)$ edge. We set $u = \langle 0, y\rangle$ with some $y \in \mathbb{Z}_2^{n-2} - \{0^{n-2}, x\}$ and let $s_h^{pq} = \langle h, pq0^{n-2}\rangle$, $u_h^{pq} = \langle h, pqy\rangle$, and $d_h^{pq} = \langle h, pqx\rangle$ for any $p, q \in \mathbb{Z}_2$,$h \in \{0, 1, 2\}$. Suppose Γ_1 is the subgraph of $BF(n-2)$ generated by $E(Q_1) \cup E(Q_2) \cup E(Q_3)$. Since Γ_1 is 0-designed, $\gamma_1^0 \circ \gamma_0^0(\Gamma_1)$ spans $BF_{0,1}^{0,0}(n)$. Similarly, let Γ_2 be the subgraph of $BF(n-2)$ generated by $E(P_1) \cup E(P_2) \cup E(P_3)$. Then $\gamma_1^j \circ \gamma_0^i(\Gamma_2)$ spans $BF_{0,1}^{i,j}(n)$. By Lemma 5, there is a hamiltonian path $\langle d_2^{01}, T_{01}, s_1^{01}\rangle$ of $BF_{0,1}^{0,1}(n)$. By Corollary 2, there is a hamiltonian path $\langle d_0^{10}, T_{10}, s_1^{10}\rangle$ of $BF_{0,1}^{1,0}(n)$. By Lemma 7, there is a hamiltonian cycle $H_{11} = \langle u_2^{11}, W_{11}, u_0^{11}, u_1^{11}, u_2^{11}\rangle$ of $BF_{0,1}^{1,1}(n) - \{s_0^{11}, s_1^{11}, s_2^{11}\}$.

Subcase 2.2.1: Assume that $ij = 10$. Let $A = \{(d_1^{00}, d_2^{01}),\ (s_1^{01}, s_0^{11}),\ (s_2^{11}, s_1^{10}),\ (u_1^{00}, u_0^{10}),\ (u_0^{00}, u_1^{10}),\ (u_1^{10}, u_2^{11}),\ (u_1^{11}, u_2^{10})\}$ and $B = \{(d_1^{00}, d_2^{00}),\ (u_0^{00}, u_1^{00}),\ (s_1^{10}, s_2^{10}),\ (u_0^{10}, u_1^{10}),\ (u_1^{10}, u_2^{10}),\ (u_1^{11}, u_2^{11})\}$. Then a 3*-container of $BF(n)$ can be formed from the subgraph generated by $(E(\gamma_1^0 \circ \gamma_0^0(\Gamma_1)) \cup E(\gamma_1^0 \circ \gamma_0^1(\Gamma_2)) \cup E(T_{01}) \cup E(H_{11}) \cup A) - B$. See Figure 12(e) for illustration.

Subcase 2.2.2: Assume that $ij = 01$. Let $A = \{(d_1^{00}, d_0^{10}),\ (s_1^{10}, s_2^{11}),\ (s_0^{11}, s_1^{01}),\ (u_2^{00}, u_1^{01}),\ (u_1^{01}, u_0^{11}),\ (u_1^{11}, u_0^{01}),\ (u_1^{00}, u_2^{01})\}$ and $B = \{(d_1^{00}, d_2^{00}),\ (u_1^{00}, u_2^{00}),\ (s_1^{01}, s_2^{01}),\ (u_0^{01}, u_1^{01}),\ (u_1^{01}, u_2^{01}),\ (u_0^{11}, u_1^{11})\}$. Then a 3*-container of $BF(n)$ can be formed from the subgraph generated by $(E(\gamma_1^0 \circ \gamma_0^0(\Gamma_1)) \cup E(\gamma_1^1 \circ \gamma_0^0(\Gamma_2)) \cup E(T_{10}) \cup E(H_{11}) \cup A) - B$. See Figure 12(f) for illustration.

Subcase 2.2.3: Assume that $ij = 11$. By Lemma 4, there is a totally scheduled hamiltonian cycle $O^{10} = \langle u_0^{10}, D_{10}, u_2^{10}, u_1^{10}, u_0^{10}\rangle$ of $BF_{0,1}^{1,0}(n)$. By

Lemma 8, there is a hamiltonian path P of $BF(n-2)$ joining $s' = \langle 0, 0^{n-2} \rangle$ to $d' = \langle 0, x \rangle$ such that P not only begins with a level-$(n-3)$ edge but also ends up with a level-0 edge. Obviously, $\gamma_1^1 \circ \gamma_0^0(P)$ joins s_0^{01} to d_2^{01}. Let $A = \{(d_1^{00}, d_2^{01}), (s_0^{01}, s_1^{11}), (u_1^{00}, u_0^{10}), (u_2^{10}, u_1^{11}), (u_0^{00}, u_1^{10}), (u_1^{10}, u_2^{11})\}$ and $B = \{(d_1^{00}, d_2^{00}), (u_0^{00}, u_1^{00}), (u_0^{10}, u_1^{10}), (u_1^{10}, u_2^{10}), (s_1^{11}, s_2^{11}), (u_1^{11}, u_2^{11})\}$. Then the subgraph Ω generated by ($E(\gamma_1^0 \circ \gamma_0^0(\Gamma_1)) \cup E(\gamma_1^1 \circ \gamma_0^1(\Gamma_2)) \cup E(O^{10}) \cup E(\gamma_1^1 \circ \gamma_0^0(P)) \cup A$) $-B$ forms a 3-container of $BF(n)$ between s and d such that each of $\{s, d\}$ is incident with only one level-0 edge. See Figure 12(g) for illustration. By Lemma 2, $V(BF(n)) - V(\Omega) = V(BF_{0,1}^{0,1}(n)) - V(\gamma_1^1 \circ \gamma_0^0(P))$. Hence, by Lemma 3, there is a 3^*-container of $BF(n)$ between s and d with the desired property. □

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